coast phase, the missile must satisfy certain altitude and flight path angle conditions dictated by the terminal guidance requirements. The case of flight in a vertical plane will be considered here. The optimal control problem considered is

$$\min_n \left[-\frac{1}{2} W_S S^2(t_f) - \int_{t_0}^{t_f} W_q q dt \right] \tag{10}$$

subject to the differential constraints

$$\dot{S} = V\cos\gamma, \dot{h} = V\sin\gamma, \dot{V} = -D/m - g\sin\gamma, \dot{\gamma} = g/V(n - \cos\gamma) \tag{11}$$

Final time t_f is given. Here S is down range distance, h altitude, V airspeed, γ flight-path angle, D aerodynamic drag, g acceleration due to gravity, m missile mass, and q dynamic pressure n is the load factor, the control variable in the present problem. W_S and W_q are weights on final range and dynamic pressure. The aerodynamic drag is modeled as

$$D = qsC_{D_0}(M) + K(M)m^2 gn^{22}/qs$$

In the above expression,, s is the reference area, C_{D_0} the zero-lift drag coefficient, K the induced drag coefficient and M is the Mach number. Load factor, the control variable, is constrained as

$$|n| \leq n\text{max}$$

The initial conditions on all the state variables are given, while only the final altitude and flight path angle are specified. The quasi-linearization scheme is next set up by linearizing the state/costate system along with the transversality condition about nominal trajectories.[3]

A computer code was written on a VAX11/750 serial machine to evaluate the quasi-linearization-integrating matrix algorithm. Figure 1 gives the results of this numerical study for the boost-glide missile trajectory problem for the following boundary conditions.

Initial conditions: $h(0) = 20000\text{m}, V(0) = 1000\text{m/s}$, $\gamma(0) = 0$ deg

Final conditions: $h(9) = 19000\text{m}$, $\gamma(9) = 0$ deg

In this particular example, a step size of one second was chosen and a third-degree Newton integrating matrix was employed. The initial guess trajectories for the states were chosen as constants. The initial guesses on costates were generated using an adjoint-control transformation similar to that given in Ref. 12. The quasi-linearization convergence factor ϵ was gradually increased from 0.001 in the first iteration to unity in the 14th iteration. The solutions given in Fig. 1 have been compared with those obtained from a nonlinear programming approach and have been found consistent.

Conclusions

The quasi-linearization alogirithm using the integrating matrix approach described in this paper reduces the nonlinear two-point boundary-value problem to a series of linear algebraic systems and is highly suited for implementation on parallel computing machines. The efficacy of this algorithm was illustrated for a boost-glide missile trajectory optimization on a serial computing machine. This approach can also be used for solving multipoint boundary-value problems and systems with state-costate discontinuities. Mechanization of this algortihm on state-of-the-art parallel processors will be of future interest.

Acknowledgment

The research for this study was supported in part by Eglin Air Force Base under SBIR Contract F8635-83-C-0480.

References

[1]Kung, H.T., "Why Systolic Architectures?," *IEEE Computer Journal*, Jan. 1982, pp. 37-46.

[2]Mead, C., and Conway, L., *Introduction to VLSI Systems*, Addison-Wesley, Reading, MA, 1980.

[3]Bryson, A.E., and Ho., Y.C., *Applied Optimal Control*, Blaisdell, Waltham, MA, 1969.

[4]Keller, H.B., "Numerical Methods of Two-Point Boundary-Value Problems," SIAM Monograph, Philadelphia, PA, 1976.

[5]Hargraves, C., Johnson, F., Parris, S., and Rettie, I., "Numerical Computation of Optimal Atmospheric Trajectories," *Journal of Guidance and Control*, Vol. 4, July-Aug. 1981, pp. 406-414.

[6]Straeter, T., "A Parallel Variable Metric Optimization Algorithm," NASA TND-7329, 1973.

[7]Chazan, D., and Miranker, W.L., "A Nongradient and Parallel Algorithm for Unconstrained Minimization," *SIAM Journal of Control*, Vol. 8, 1970, pp. 207-217.

[8]Travassos, R., and Kaufman, H., "Parallel Algorithms for Solving Nonlinear Two-Point Boundary-Value Problems Which Arise in Optimal Control," *Journal of Optimization Theory and Applications*, Vol. 30, Jan. 1980, pp. 53-71.

[9]Lehman, L.L., "Hybrid State Vector Methods for Structural Dynamic and Aeroelastic Boundary Value Problems," NASA CR-3591, Aug. 1982.

[10]Heller, D., "A Survey of Parallel Algorithms in Numerical Linear Algebra," *SIAM Review*, Vol. 20, Oct. 1978, pp. 740-777.

[11]Duff I.S. (Editor), *Sparse Matrices and Their Uses*, Academic Press, New York, 1981.

[12]Dixon, L.C.W., and Bartholomew-Biggs, M.C., "Adjoint-Control Transformations for Solving Practical Optimal Control Problems," *Optimal Control Applications and Methods*, Vol. 2, 1981, pp. 365-381.

The Principal Minor Test for Semidefinite Matrices

John E. Prussing*
University of Illinois at Urbana-Champaign
Urbana, Illinois

POSITIVE definite and positive semidefinite (also called nonnegative definite) real quadratic and Hermitian forms play important roles in many control and dynamics applications. A quadratic form $q = x^T Ax$ and its associated real symmetric $n \times n$ matrix A are termed positive definite if $q > 0$ for all $x \neq 0$, and positive semidefinite if $q \geq 0$ for all x. The eigenvalues of a positive definite (semidefinite) matrix are all positive (nonnegative).

Received June 14, 1985.

*Professor, Department of Aeronautical and Astronautical Engineering. Associate Fellow AIAA.

A practical test for positive definiteness that does not require explicit calculation of the eigenvalues is the principal minor test. The kth leading principal minor is the determinant formed by deleting the last $n-k$ rows and columns of the matrix. A necessary and sufficient condition that a symmetric $n \times n$ matrix be positive definite is that all n leading principal minors Δ_k are positive.[1]

However, the analogous statement that a necessary and sufficient condition that a matrix be positive semidefinite is that all n leading principal minors are nonnegative is not true, yet this statement is found in some textbooks and reference books. Greenwood[2] states that if one or more of the leading principal minors are zero, but none are negative, then the matrix is positive semidefinite. This is disproved by the examples in this Note. Brogan[3] states that a test for positive semidefiniteness is that all of the leading principal minors are nonnegative, which incorrectly implies that if this test is satisfied, the matrix is positive semidefinite. Wiberg[4] provides the analogous condition for a Hermitian matrix, namely, that it is positive semidefinite, if, and only if, all of the leading principal minors are nonnegative.

As a trivial example consider the matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

Both leading principal minors are zero and hence nonnegative, but the matrix is obviously not positive semidefinite. One eigenvalue is zero, the other is -1.

Another example is the 3×3 symmetric matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & a \end{bmatrix} \tag{2}$$

The leading principal minors are nonnegative ($\Delta_1 = 1$, $\Delta_2 = \Delta_3 = 0$), but the matrix is not positive semidefinite. This can be verified by calculating the value of the quadratic form $q = x^T A x$, where $x^T = [x_1 x_2 x_3]$ as $q = (x_1 + x_2 + x_3)^2 + (a-1)x_3^2$. By inspection, one can guarantee $q \geq 0$ for all x only if $a \geq 1$. For example, if $a < 1$, vectors in the plane $x_1 + x_2 + x_3 = 0$ yield $q < 0$. Note, however, that the values of the leading principal minors are independent of the value a.

It is easily shown that all of the leading principal minors of a positive semidefinite matrix are nonnegative, for example, by considering vectors x having the last $n-k$ elements equal to zero. Thus, the condition that $\Delta_k \geq 0$ is apparently a necessary but not a sufficient condition for positive semidefiniteness. In order to state a condition that is also sufficient, one must consider principal minors D_k formed by deleting *any* $n-k$ rows and corresponding columns. The correct necessary and sufficient condition is that all possible principal minors are nonnegative.

As an example, consider the matrix in Eq. (1). If one calculates the principal minors D_k formed by deleting the first rather than last $n-k$ rows and columns, one finds that $D_1 = -1$ and $D_2 = 0$, which clearly violates the condition. In the same manner, the principal minors of the matrix in Eq. (2) are $D_1 = a$, $D_2 = a-1$, and $D_3 = 0$, satisfying the condition only if $a \geq 1$.

For an $n \times n$ matrix that are $\binom{n}{k}$ principal minors D_k, where $\binom{n}{k}$ is the binomial coefficient $n!/(n-k)!k!$. The total number of principal minors is then

$$m = \sum_{k=1}^{n} \binom{n}{k} = 2^n - 1 \tag{3}$$

Thus, $m = 7$ for $n = 3$, and $m = 15$ for $n = 4$, indicating that appreciably more computation may be required to determine the value of all possible principal minors. However, if all n leading principal minors of a matrix are nonnegative, with one or more having a zero value, all m principal minors must be shown to be nonnegative in order to guarantee that the matrix is positive semidefinite.

References

[1]Hildebrand, F.B., *Methods of Applied Mathematics*, Prentice-Hall, Englewood Cliffs, NJ, 1961, p. 50.

[2]Greenwood, D.T., *Principles of Dynamics*, Prentice-Hall, Englewood Cliffs, NJ, 1965, p. 452.

[3]Brogan, W.L., *Modern Control Theory*, Quantum Publishers, New York, 1974, p. 134.

[4]Wiberg, D.M., *State Space and Linear Systems*, Schaum Outline Series, McGraw-Hill Book Co., New York, 1971, p. 77.

Gravity Gradient Torque for an Arbitrary Potential Function

David R. Glandorf*
Lockheed Engineering and Management Services Company
Houston, Texas

Nomenclature

C	$= C_a$ to C_s transformation matrix
C_a	= attracting body coordinate system
C_s	= spacecraft coordinate system
dm	= differential mass
$\boldsymbol{F}_g$	= gravitational force
$g \triangleq (g_x, g_y, g_z)^T$	= components of $\boldsymbol{g}$ in C_a
$\boldsymbol{g}$	= gravitational acceleration
G	= gravity gradient dyadic
G	= gravity gradient matrix in C_a
I	$= 3 \times 3$ identity matrix
I_x, I_y, I_z	= spacecraft moments of inertia
I_{xy}, I_{xz}, I_{yz}	= spacecraft products of inertia
J	= spacecraft inertia dyadic
J	= components of J in C_s
m	= spacecraft mass
r	= magnitude of $\boldsymbol{R}$
$R \triangleq (x,y,z)^T$	= components of $\boldsymbol{R}$ in C_a
$\boldsymbol{R}$	= location of dm
$\boldsymbol{R}_c$	= location of spacecraft mass center
$U \triangleq (U_x, U_y, U_z)^T$	= components of U in C_a
$\boldsymbol{U}$	= unit vector in the direction of $\boldsymbol{R}$
λ_i	= eigenvalues of G, $i = 1,2,3$
μ	= gravitational parameter of attracting body
$\rho \triangleq (\rho_x, \rho_y, \rho_z)$	= components of $\boldsymbol{\rho}$ in C_s
$\boldsymbol{\rho}$	= location of dm relative to spacecraft mass center
$\tau_{gg} \triangleq (\tau_{ggx}, \tau_{ggy}, \tau_{ggz})^T$	= components of $\boldsymbol{\tau}_{gg}$ in C_s
$\boldsymbol{\tau}_g$	= gravitational moment
$\boldsymbol{\tau}_{gg}$	= gravity gradient torque
ϕ	= gravitational potential
ψ_i	= components of $\boldsymbol{\psi}_i$ in C_s, $i = 1,2,3$
$\hat{\psi}_i$	= components of $\boldsymbol{\psi}_i$ in C_a, $i = 1,2,3$
$\boldsymbol{\psi}_i$	= eigenvectors of G, $i = 1,2,3$

Received April 1, 1985; revision received July 16, 1985.

*Advanced Systems Engineering Specialist. Member AIAA.