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Engineering Notes

Simplified Conjugate Point Procedure

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I. Introduction

IN OPTIMAL control problems satisfying first-order necessary conditions (NCs) (for example, in Bryson and Ho [1]), is typically used to determine an optimal solution. But a solution to the first-order NC is not an optimal solution if it contains a conjugate point, because that violates the second-order Jacobi NC and there exists a neighboring solution of lower cost. In the past, testing for a conjugate point was usually not done because of the difficulty in doing so. Because of this, some conjugate points may have been overlooked, resulting in nonoptimal solutions. In this Note, a relatively new simplified procedure to test for a conjugate point is illustrated on a simple example problem.

A classical geometrical example of a conjugate point occurs in the problem of the minimum-distance path between two points on a sphere. A great circle connecting the points satisfies the first-order NC, but a conjugate point exists at a point diametrically opposite the initial point. A solution containing this conjugate point is not minimizing and there exists a neighboring path that is shorter (lower cost). In Jo and Prussing [2], a continuous-thrust trajectory in the two-body problem is presented that satisfies all the first-order NCs, but contains a conjugate point, and is therefore nonoptimal. A neighboring solution is determined that has a 14% lower cost.

II. Analysis

This Note presents a simple example of a conjugate point in the two-dimensional problem of the minimum time path of a vehicle moving at constant speed in the x_1–x_2 plane from a point to a circle. Using the notation of Bryson and Ho [1], for a time interval $0 \le t \le t_f$, the cost is defined to be $J = \phi[\boldsymbol{x}(t_f), t_f] = t_f$ and the initial point is defined in terms of the two state variables as a point on the x_1 axis:

$$\boldsymbol{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \tag{1}$$

The equation of motion (for a unit vehicle speed) is

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \tag{2}$$

where θ is the heading angle (the control variable). The terminal constraint is the unit circle:

$$\psi[\boldsymbol{x}(t_f), t_f] = \frac{1}{2}\left[x_1^2(t_f) + x_2^2(t_f) - 1\right] = 0 \tag{3}$$

Consider the case in which $0 < a < 1$ in Eq. (2) (i.e., the initial point is inside the circle, but not at the origin).

III. First-Order Necessary Conditions

To apply the first-order NC [1], the Hamiltonian function is defined as

$$H(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{\lambda}, t) = L + \boldsymbol{\lambda}^T \boldsymbol{f} = \lambda_1 \cos\theta + \lambda_2 \sin\theta \tag{4}$$

where $L \equiv 0$ in this example. An augmented terminal function is defined as

$$\boldsymbol{\Phi}[\boldsymbol{x}(t_f), t_f, \nu] \equiv \varphi + \nu\psi = t_f + \frac{\nu}{2}\left[x_1^2(t_f) + x_2^2(t_f) - 1\right] \tag{5}$$

Applying the first-order NC,

$$\frac{\partial H}{\partial \theta} = 0 = -\lambda_1 \sin\theta + \lambda_2 \cos\theta \tag{6}$$

which yields

$$\tan\theta = \frac{\lambda_2}{\lambda_1} \tag{7}$$

Next,

$$\dot{\lambda}^T = -\frac{\partial H}{\partial \boldsymbol{x}} = 0^T \tag{8}$$

indicating a constant $\boldsymbol{\lambda}(t)$, which implies a constant heading angle θ. The boundary condition for Eq. (8) is

$$\lambda^T(t_f) = \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{x}(t_f)} = \nu[x_1(t_f) \;\; x_2(t_f)] \tag{9}$$

The solution to Eq. (2) for constant θ evaluated at the final time is

$$x_1(t_f) = a + t_f \cos\theta \tag{10a}$$

$$x_2(t_f) = t_f \sin\theta \tag{10b}$$

And for a constant θ, Eqs. (7–10b) yield

$$\tan\theta = \frac{x_2(t_f)}{x_1(t_f)} = \frac{t_f \sin\theta}{(a + t_f \cos\theta)} \tag{11}$$

which implies that $\sin\theta = 0$ for $a \ne 0$. And thus the values of θ that satisfy the NC are zero and π, for which $x_2(t_f) = 0$ and $x_1(t_f) = 1$ or -1, respectively.

Because the final time t_f is unspecified, there is an additional NC, $\boldsymbol{\Omega} = \mathrm{d}\boldsymbol{\Phi}/\mathrm{d}t_f + L(t_f) = 0$, which can be calculated using $\partial\boldsymbol{\Phi}/\partial t_f + H(t_f) = 0 = 1 + \nu x_1(t_f)\cos\theta$, which yields $\nu = -1$ for both $\theta = 0$ and π.

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IV. Second-Order Conditions

A procedure for applying second-order NCs and sufficient conditions (SCs) is summarized and illustrated in Prussing and Sandrik [3]. This analysis is based on earlier work by Wood [4,5] and by Breakwell and Ho [6]. One second-order NC is the Legendre–Clebsch condition $H_{\theta\theta} \geq 0$, and part of the SC is the convexity condition, which is the strengthened Legendre–Clebsch condition $H_{\theta\theta} > 0$. From Eq. (4),

$$H_{\theta\theta} = -\lambda_1 \cos\theta - \lambda_2 \sin\theta \tag{12}$$

and from Eq. (9) for $\theta = 0$, $\lambda_1 = \nu x_1(t_f) = -1$; and for $\theta = \pi$, $\lambda_1 = \nu x_1(t_f) = 1$, and so $H_{\theta\theta}$ is equal to one for both cases. Both Legendre–Clebsch conditions are satisfied.

A second-order NC and another part of the SC is the Jacobi no-conjugate-point condition. Part of the procedure for determining a conjugate point is the terminal constraint nontangency condition $\mathrm{d}\psi/\mathrm{d}t_f \neq 0$. Because ψ in Eq. (3) is not an explicit function of t_f, the derivative is calculated as

$$\begin{aligned}\frac{\mathrm{d}\psi}{\mathrm{d}t_f} &= \frac{\partial\psi}{\partial x_1(t_f)}\dot{x}_1(t_f) + \frac{\partial\psi}{\partial x_2(t_f)}\dot{x}_2(t_f) \\ &= x_1(t_f)\cos\theta + x_2(t_f)\sin\theta = 1\end{aligned} \tag{13}$$

for both $\theta = 0$ and $\theta = \pi$.

For this example problem, there is a single terminal constraint, and so the procedure is simpler than for multiple terminal constraints [3]. One needs to determine a 4×4 transition matrix $\Theta(t, t_f)$ that satisfies

$$\dot{\Theta}(t, t_f) = P(t)\Theta(t, t_f); \qquad \Theta(t_f, t_f) = \boldsymbol{I}_4 \tag{14}$$

where

$$P = \begin{bmatrix} A_1 & -A_2 \\ -A_0 & -A_1^T \end{bmatrix} \tag{15}$$

and the 2×2 partitions in Eq. (15) depend on the equation of motion Eq. (2), and the Hamiltonian of Eq. (4):

$$A_0 = H_{xx} - H_{x\theta}H_{\theta\theta}^{-1}H_{\theta x} \tag{16a}$$

$$A_1 = \boldsymbol{f}_x - \boldsymbol{f}_\theta H_{\theta\theta}^{-1}H_{\theta x} \tag{16b}$$

$$A_2 = \boldsymbol{f}_\theta H_{\theta\theta}^{-1}\boldsymbol{f}_\theta^T \tag{16c}$$

In the example problem $\boldsymbol{f}_x = 0_2$ (a 2×2 zero matrix) and $H_{\theta x} = 0_2$, and so $A_o = A_1 = 0_2$. The matrix A_2 is (using $\sin\theta = 0$)

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{17}$$

for both $\theta = 0$ and π. Therefore, for the 4×4 P matrix in Eq. (15), all the elements are zero except $P_{24} = -1$ and the corresponding 4×4 transition matrix in Eq. (14) is

$$\Theta(t, t_f) = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \tag{18a}$$

where the 2×2 partitions are

$$\Theta_{11} = \Theta_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{18b}$$

$$\Theta_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{18c}$$

and

$$\Theta_{12} = \begin{bmatrix} 0 & 0 \\ 0 & t_f - t \end{bmatrix} \tag{18d}$$

Finally, a conjugate point exists if a certain 2×2 matrix $X(t)$ is singular, namely,

$$X(t) = \Theta_{11} + \Theta_{12}S_F \tag{19}$$

Note that $X(t_f) = \boldsymbol{I}_2$, based on the boundary condition in Eq. (14). The matrix S_F is given by Eq. (14) in [3] and is a complicated expression, but in this simple example it reduces to $S_F = \nu\boldsymbol{I}_2 = -\boldsymbol{I}_2$. Evaluating Eq. (19) using Eqs. (18b) and (18d),

$$\begin{aligned}X(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & t_f - t \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 + t - t_f \end{bmatrix}\end{aligned} \tag{20}$$

and

$$\det X(t) = 1 + t - t_f \tag{21}$$

At this point, the two cases $\theta = 0$ and $\theta = \pi$ must be considered separately.

1) For $\theta = 0$, $x_1(t_f) = 1$ and from Eq. (10a) $t_f = 1 - a$. From Eq. (21), det $X(t) = 0$ for $t = -a$, but that is a negative value and is outside the time interval [0, $1 - a$] for this solution. Therefore, no conjugate point exists for $\theta = 0$.

2) For $\theta = \pi$, $x_1(t_f) = -1$ and from Eq. (10a) $t_f = 1 + a$. From Eq. (21), det $X(t) = 0$ for $t = a$, which is the time that the vehicle crosses the origin. Therefore, a conjugate point exists at the origin for $\theta = \pi$, the solution is not the minimum time solution, and there exists a neighboring solution that requires less time.

Such a lower cost neighboring solution is obtained by simply bypassing the origin and avoiding the conjugate point. Consider a straight line path from the initial point to the terminal constraint with a final time $\hat{t}_f$ Let $x_2(\hat{t}_f)$ be a small positive number ε rather than zero. Then, from Eq. (3), $x_1^2(\hat{t}_f) = 1 - \varepsilon^2$, that so $x_1(\hat{t}_f) = \sqrt{(1 - \varepsilon^2)}$ which is approximately $1 - (1/2)\varepsilon^2$. The time $\hat{t}_f$ to get to this point (which is equal to the distance traveled), is given by

$$\hat{t}_f^2 = \varepsilon^2 + \left(1 + a - \frac{1}{2}\varepsilon^2\right)^2 \tag{22a}$$

which is approximately (small ε):

$$\varepsilon^2 + (1 + a)^2 - \varepsilon^2(1 + a) = (1 + a)^2 - \varepsilon^2 a < (1 + a)^2 \tag{22b}$$

Therefore, $\hat{t}_f < 1 + a$ and is a smaller time than $t_f = 1 + a$ on the stationary solution containing the conjugate point.

This example is very simple, but demonstrates the basic idea. The procedure is more complicated when there are multiple terminal constraints. Prussing and Sandrick [3] provide a concise summary of the procedure for more complicated cases and show a more complicated example problem.

V. Conclusions

In this Note, a simplified procedure for determining conjugate points is described and illustrated by a simple geometrical example. Testing for conjugate points is important because a solution satisfying all the first-order necessary conditions is nonoptimal if it contains a conjugate point and there exists a neighboring solution of lower cost. As shown by the example here, even a very simple problem can have a conjugate point and whether one exists is not obvious at the outset. An advantage in testing for a conjugate point is that, if none exists, and the other SCs are satisfied, the solution is a local optimum.

References

[1] Bryson, A. E., Jr., and Ho, Y.-C., *Applied Optimal Control*, Hemisphere, Washington, D.C., 1975, Chap. 2.

[2] Jo, J.-W., and Prussing, J. E., "Procedure for Applying Second-Order Conditions in Optimal Control Problems," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 2, March–April 2000, pp. 241–250. doi:10.2514/2.4546

[3] Prussing, J. E., and Sandrik, S. L., "Second-Order Necessary Conditions and Sufficient Conditions Applied to Continuous-Thrust Trajectories," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 4, July–Aug. 2005, pp. 812–816. doi:10.2514/1.4040

[4] Wood, L. J., "Sufficient Conditions for a Local Minimum of the Bolza Problem with a Scalar Terminal Point Constraint," *Advances in the Astronautical Sciences*, Vol. 76, Pt. 3, Aug. 1991, pp. 2053–2072; also American Astronautical Soc. Paper 91-449, Aug. 1991.

[5] Wood, L. J., "Sufficient Conditions for a Local Minimum of the Bolza Problem with Multiple Terminal Point Constraints," American Astronautical Soc. Paper 1991-450, Aug. 1991.

[6] Breakwell, J. V., and Ho, Y.-C., "On the Conjugate Point Condition for the Control Problem," *International Journal of Engineering Science*, Vol. 2, No. 6, 1965, pp. 565–579. doi:10.1016/0020-7225(65)90037-6