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Engineering Notes

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Constant Radial Thrust Acceleration Redux

John E. Prussing* and Victoria Coverstone-Carroll†
University of Illinois at Urbana–Champaign,
Urbana, Illinois 61801-2935

Introduction

IN 1953 Tsien[1] analyzed escape from circular orbit in the two-body problem using a constant radial or circumferential thrust acceleration. More recent results on this problem have been obtained by Battin[2] and by Boltz[3,4] for a slightly different thrust model. The radial case is revisited here, providing more analytical results and a novel application. The approach presented uses the concept of a potential energy well to determine 1) under what conditions escape will occur and 2) a closed-form expression for the amplitude of the radial oscillation if escape does not occur. In addition, the concept of utilizing a constant radial thrust acceleration to shift the radius of a circular orbit of specified period (or to shift the period for a specified radius) is introduced and analyzed.

Potential Well

A constant radial thrust acceleration a_r can be included in the two-body problem by simply adding a term to the potential energy

$$V(r) = -\mu/r - a_r r \tag{1}$$

The first term in Eq. (1) represents the inverse-square gravitational acceleration and the second term the constant radial thrust acceleration. The negative of the gradient of V yields the total applied acceleration. The Lagrangian L of the system is formed by subtracting the potential energy V from the kinetic energy per unit mass T:

$$L = T - V = \tfrac{1}{2}[\dot{r}^2 + (r\dot{\theta})^2] + \mu/r + a_r r \tag{2}$$

Lagrange's equations then yield the results that

$$\ddot{r} - r\dot{\theta}^2 = -\mu/r^2 + a_r \tag{3}$$

along with two integrals of the motion. The first of these is because θ is an ignorable coordinate [$\partial L/\partial\theta = 0$ in Eq. (2)], which yields

$$\frac{\partial L}{\partial\dot{\theta}} = r^2\dot{\theta} = \text{const} \equiv h \tag{4}$$

where h is the angular momentum per unit mass. It is conserved because both the gravitational acceleration and the thrust acceleration are radial and apply no moment to the system.

The second integral of the motion is the Jacobi integral K, which is constant because $\partial L/\partial t = 0$. Because the kinetic energy term in Eq. (2) is a homogeneous quadratic function of the generalized velocities $\dot{r}$ and $\dot{\theta}$, the Jacobi integral is equal to the total energy of the system

$$K = T + V = \tfrac{1}{2}[\dot{r}^2 + (r\dot{\theta})^2] - \mu/r - a_r r = \text{const} \tag{5}$$

If Eq. (4) is used to eliminate the variable $\dot{\theta}$ in Eq. (5) the result is

$$K = \tfrac{1}{2}\dot{r}^2 + h^2/2r^2 - \mu/r - a_r r = \text{const} \tag{6}$$

It can be seen from Eq. (6) that the problem has been reduced to an equivalent one-dimensional (radial) one with a kinetic energy

$$T_r = \tfrac{1}{2}\dot{r}^2 \tag{7}$$

and a potential energy function

$$V_r = h^2/2r^2 - \mu/r - a_r r \tag{8}$$

which can be interpreted as representing a nonlinear radial spring.

The value of the constant Jacobi integral K is determined by initial conditions. If the spacecraft is initially in a circular orbit of radius r_0,

$$h^2 = \mu r_0 \tag{9}$$

and Eq. (8) becomes

$$V_r = \mu r_0/2r^2 - \mu/r - a_r r \tag{10}$$

along with [because $T_r(r_0) = 0$]

$$K = V_r(r_0) = -\mu/2r_0 - a_r r_0 \tag{11}$$

If Eqs. (9) and (11) are combined with Eq. (6), the result is identical with Eq. (8.94) of Ref. 2. However, here the interpretation and use of this information is different. The problem of escaping the center of attraction starting in a circular orbit will be analyzed in terms of escaping a potential energy well.

Figure 1 shows a general case of a potential well. The kinetic energy T_r is shown as the difference between the constant total energy K and the potential energy function $V_r(r)$. For the function shown, the spacecraft (starting at the initial circular radius r_0) will be trapped in the well, defined by $r_0 \le r \le R$, with the endpoints characterized by the radial kinetic energy being zero. That the slope $V_r'(r_0) < 0$ implies that the initial radial force is positive (because the force is equal to $-V_r'$), forcing the spacecraft to larger radius. That $V_r'(R) > 0$ implies that the force at maximum radius is negative, forcing the spacecraft back to smaller values of the radius. These restoring forces at the ends of the well result in a radial oscillation.

As is evident in Eq. (10), the shape of the potential energy function depends strongly on the value of the radial thrust acceleration a_r, as does the value of the Jacobi integral in Eq. (11). As will be shown, for a sufficiently large value of a_r escape will occur, but for smaller values the spacecraft will oscillate in a potential well. Characteristic of a nonlinear oscillator, the period is a function of the amplitude and is analyzed in detail in Ref. 2.

To examine different values of radial thrust acceleration, it is convenient to define a nondimensional variable α that is equal to the ratio of the radial thrust acceleration to the inverse-square gravitational acceleration at the initial circular orbit radius r_0:

$$\alpha \equiv \frac{a_r r_0^2}{\mu} \tag{12}$$

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*Professor, Department of Aeronautical and Astronautical Engineering. Fellow AIAA.

†Assistant Professor, Department of Aeronautical and Astronautical Engineering. Senior Member AIAA.

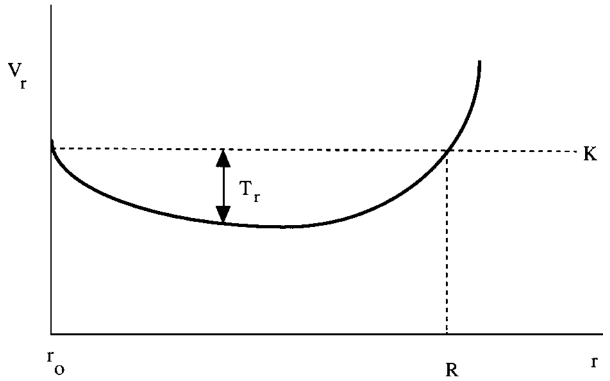

Fig. 1 Potential energy well.

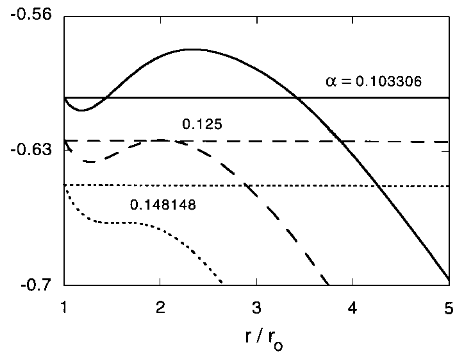

Fig. 2 Three example potential energy functions.

The upper curve in Fig. 2 shows the potential function $V_r(r)$ of Eq. (8) for a value of α corresponding to the numerical results reported in Ref. 2, namely, $\alpha = 1/9.68 = 0.103306$ [corresponding to $\beta = (8\alpha)^{-1/2} = 1.1$ in the notation of Ref. 2]. The numerically integrated orbit spirals out to a radius of $1.41183335r_0$, then spirals back in to its initial radius r_0 and continues to oscillate. This is consistent with the upper curve in Fig. 2, which shows the corresponding limits on the potential energy well, defined by the two leftmost intersections of the potential energy function with the horizontal solid line, representing the constant value K of the total energy for that value of α. The two other potential functions shown in Fig. 2 are for larger values of α and will be discussed later.

Several important properties of the potential function $V_r(r)$ are helpful in the analysis. From Eq. (10) it is evident that

$$\frac{\partial V_r}{\partial a_r} = -r < 0 \tag{13}$$

indicating that the value of V_r decreases with increasing a_r, by an amount proportional to the value of r. Other important properties are evident from Eq. (10), namely, that

$$V_r \to +\infty \quad \text{in the limit as} \quad r \to 0 \tag{14}$$

and (for $a_r > 0$)

$$V_r \to -\infty \quad \text{in the limit as} \quad r \to +\infty \tag{15}$$

(For $a_r < 0$, $V_r \to +\infty$, indicating that escape is impossible.)

Another important property is its curvature

$$V_r'' \equiv \frac{d^2 V_r}{dr^2} = \frac{\mu(3r_0 - 2r)}{r^4} \tag{16}$$

which changes sign at $r = 1.5r_0$ with

$$V_r'' > 0 \quad \text{if} \quad r < 1.5r_0 \tag{17a}$$

$$V_r'' < 0 \quad \text{if} \quad r > 1.5r_0 \tag{17b}$$

This property is evident for all three of the potential functions shown in Fig. 2.

Escape will occur if the value of a_r is large enough that the maximum value of V_r for $r > r_0$ is less than K. The critical value of a_r is determined by setting V_r equal to K and requiring V_r' to be zero. From Eqs. (10) and (11) a cubic equation can be written in terms of α and $\xi \equiv r/r_0$ as

$$2\alpha\xi^3 - (2\alpha + 1)\xi^2 + 2\xi - 1 = 0 \tag{18}$$

and the condition $V_r' = 0$ yields

$$\alpha\xi^3 - \xi + 1 = 0 \tag{19}$$

Solving Eq. (19) for α and substituting into Eq. (18) yield

$$\xi^3 - 4\xi^2 + 5\xi - 2 = 0 \tag{20}$$

One root of Eq. (20) is $\xi = 2$ ($r = 2r_0$), which yields the critical value $\alpha_{cr} = \frac{1}{8}$ in Eq. (19). The other two roots are equal and are given by $\xi = 1$, corresponding to $\alpha = a_r = 0$, representing no thrust ($\xi = 1$ with no thrust corresponds to the initial circular orbit, for which the potential function has a global minimum).

The condition for escape to occur is then simply

$$\alpha > \alpha_{cr} = \tfrac{1}{8} = 0.125 \tag{21}$$

in agreement with Refs. 1 and 2.

Escape occurs when the total orbital energy (without the thrust term in the potential energy) reaches the value zero. From Eq. (6) this occurs at the value r_e of the radius, where

$$K = 0 - a_r r_e \tag{22}$$

Using Eq. (11) for the value of K, the value of r_e is determined to be

$$r_e = r_0[1 + (1/2\alpha)] \tag{23}$$

in agreement with Eq. (8.95) of Ref. 2. For the escape condition $\alpha > \frac{1}{8}$, Eq. (23) yields $r_e < 5r_0$.

The middle curve in Fig. 2 shows the potential function V_r for the critical value $\alpha_{cr} = \frac{1}{8} = 0.125$. Note the upper limit on the potential well is at $r = 2r_0$, corresponding to the root of Eq. (20) discussed earlier. The lowest curve in Fig. 2 shows the potential function for the higher (escape) value $\alpha = \frac{4}{27} = 0.148148$ for which, from Eq. (19), the slope at the inflection point $V_r'(1.5r_0) = 0$ and no potential well exists. From Eq. (23) the value of r_e is determined to be $4.375r_0$.

For values of α less than or equal to α_{cr}, no escape occurs. The magnitude of the resulting radial oscillation is given by Eq. (18) alone. Because Eq. (18) is satisfied by $\xi = 1$ for all values of α (as it must be) the remaining roots must satisfy

$$2\alpha\xi^2 - \xi + 1 = 0 \tag{24}$$

These roots are real for $\alpha \le \alpha_{cr} = \frac{1}{8}$, and the smaller root is equal to

$$\xi = \frac{1 - (1 - 8\alpha)^{\frac{1}{2}}}{4\alpha} \tag{25}$$

This smaller root represents the right end of the potential well for $\alpha > 0$, as seen in the upper curve of Fig. 2. (For $\alpha < 0$ it represents the left end of the well, which is at a radius less than r_0 with the right end being at r_0.) Equation (25) provides a closed-form expression for the amplitude of the radial oscillation. For the numerical case reported in Ref. 2 and mentioned earlier, $\alpha = 1/9.68 = 0.103306$, for which Eq. (25) yields the exact value $\xi = 2.42 - (1.0164)^{1/2}$, which has the decimal representation 1.41183335.

For values of α greater than α_{cr}, the two roots of Eq. (25) are complex, indicating escape (no potential well), as shown in the lowest curve in Fig. 2.

Shifted Circular Orbits

The local minimum points of the upper two potential energy curves in Fig. 2 represent the locations of stable circular orbits at radii larger than r_0. This is because an outward radial thrust acceleration has the effect of decreasing the value of the local gravitational acceleration. An inward radial thrust ($\alpha < 0$) increases the effective value of the gravitational acceleration, resulting in a stable circular orbit at a radius less than r_0. The upper two curves in Fig. 2 both exhibit a single local minimum at a value of $r > r_0$. The lowest curve is the limiting case, where the local minimum ceases to exist.

For a circular orbit of radius r_c, the orbital speed is $v_c = r_c\dot{\theta}_c$ and $\dot{r}_c = \ddot{r}_c = 0$. Equation (3) then yields

$$v_c^2 = (\mu/r_c) - a_r r_c \tag{26}$$

The shifted circular orbit period in the presence of the radial thrust can then be determined simply as

$$P = \frac{2\pi r_c}{v_c} = 2\pi\left[\frac{r_c^3}{\mu}\right]^{\frac{1}{2}} \frac{1}{\left[1-\left(a_r r_c^2/\mu\right)\right]^{\frac{1}{2}}}$$

$$= \frac{P_c}{\left[1-\left(a_r r_c^2/\mu\right)\right]^{\frac{1}{2}}} = \frac{P_c}{\left[1-\alpha(r_c/r_0)^2\right]^{\frac{1}{2}}} \tag{27}$$

where P_c is the no-thrust circular orbit period at radius r_c.

From Eq. (27) it is evident that $P > P_c$ if $\alpha > 0$ and $P < P_c$ if $\alpha < 0$. This means that for $\alpha > 0$ a shifted circular orbit of period P will have a radius smaller than the no-thrust circular orbit with the same period. As a specific example of a shifted orbit, consider a geosynchronous orbit. Take the maximum-shift value of $\alpha = \frac{4}{27} = 0.148148$ corresponding to a shifted circular orbit at $r_c = 1.5r_0$. (This orbit is actually unstable, and so a slightly smaller value of α would need to be used in practice.) For $P = 23.934$ h, Eq. (25) yields a value of $P_c = 19.542$ h, corresponding to a shifted circular orbit of radius of 36,834 km. This is 5330 km lower than the conventional (no-thrust) geosynchronous radius of 42,164 km, but the period is the same. In terms of orbital altitude, this represents a 15% decrease. The level of constant radial thrust acceleration necessary to maintain this orbit is about 0.1 m/s^2 or approximately 10^{-2} g. This lower altitude geosynchronous orbit may provide some advantages. A shifted 1-year heliocentric orbit requires a much lower thrust acceleration of 0.0026 m/s^2.

Conclusions

The use of constant radial thrust acceleration on a circular orbit in the two-body problem is revisited. The concept of a potential energy well provides an explanation of why a radial oscillation occurs for values of thrust acceleration below a critical value and escape occurs for higher thrust levels. This critical value of thrust acceleration is determined, along with a formula for the value of the radius at which escape energy is achieved and a formula for the amplitude of the radial oscillation if escape does not occur. The concept of using a constant radial thrust acceleration to shift the radius of a circular orbit of specified period (or shift the period for a specified radius) is shown to be feasible and potentially useful for Earth-orbit and heliocentric applications.

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Two-Degree-of-Freedom $\mathcal{H}_\infty$ Robust Controller for a Flexible Missile

K. Koshy George* and M. Seetharama Bhat†
Indian Institute of Science, Bangalore 560 012, India

Introduction

THE introduction of long, slender missiles, such as the Vanguard, the Redstone, etc., has resulted in a rather acute problem of structural flexibility. These missiles are made as light as possible due to the limited thrust available, and hence structural rigidity is sacrificed. The satisfactory performance of control systems is crucially dependent on an accurate representation of the vehicle's elastic deformation under prescribed forces. This is because under adverse conditions, the control system can reinforce the amplitudes of the local elastic distortions, possibly leading to the structural failure of the vehicle. The Lark missile, built in the United States, and the ASLV-D2 launch vehicle, built by the Indian Space Research Organization, are examples of vehicles that exhibited such a control and structure interaction. Therefore, it is important that the effect of structural flexibility be directly controlled. However, because there is uncertainty in the representation of the elastic modes, robust controllers are to be designed.

We design an $\mathcal{H}_\infty$ robust two-degree-of-freedom (DOF) robust pitch plane flight control system for a flexible missile. The cascade compensator $F(s)$ is designed using a multicriteria vector optimization technique based on the minimization of the Kreisselmeier function,[1] and the feedback compensator $K(s)$ is designed using the $\mathcal{H}_\infty$ design algorithm given by Glover and Doyle.[2] The compensator $K(s)$ is placed in the feedback path to simplify the computation of the controller.[3] A two-DOF structure is chosen to improve the time-domain performance of the closed-loop system. The design is presented in the next section.

Controlling the Pitch Plane Dynamics

The flexible missile considered is a long, slender, ballistic missile. The missile is roll stabilized a priori, and hence only the pitch plane dynamics is considered. The linearized equations of the dynamics is described in terms of the pitch angle θ, the pitch angle rate q, and the angle of attack α. The effect of flexibility is described in terms of the generalized coordinates q_j. For the example considered, data for the first three bending modes are available, and satisfy $\ddot{q}_j + 2\zeta_j\dot{q}_j\omega_j + \omega_j^2 q_j = Q_j/M_j$, $1 \le j \le 3$, where ζ_j, ω_j, Q_j, and M_j are, respectively, the damping ratio, the natural frequency, the generalized force, and the generalized mass associated with the jth bending mode. The natural frequencies are 33.94, 90.73, and 163.44 rad/s with respective damping ratios 0.00646, 0.00426, and 0.00404. We assume a second-order actuator model.

We define the plant state vector as

$$\boldsymbol{x}_p = (\theta \quad q \quad \alpha \quad \delta \quad \dot{\delta} \quad q_1 \quad \dot{q}_1 \quad q_2 \quad \dot{q}_2 \quad q_3 \quad \dot{q}_3)^T$$

where δ and $\dot{\delta}$ are the states associated with the actuator. Thus, the plant model can be represented by $y_p(s) = P(s)u_p(s)$, where u_p is the control input, and $y_p = (\theta_m \quad q_m)^T$, the measured pitch angle and pitch rate, are given by

$$\theta_m = \theta + \sum_{j=1}^{3}\phi'_{\mathrm{pg}_j} q_j, \qquad q_m = q + \sum_{j=1}^{3}\phi'_{\mathrm{rg}_j}\dot{q}_j$$

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*Research Scholar, Department of Aerospace Engineering; currently Postdoctoral Fellow, Automatic Control Research Center, Seoul National University, Seoul 151-742, Republic of Korea.

†Associate Professor, Department of Aerospace Engineering. E-mail: msbdcl@aero.iisc.ernet.in.