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Procedure for Applying Second-Order Conditions in Optimal Control Problems

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A recent advance in sufficient conditions for a weak local minimum in the Bolza optimal control problem is used to develop a practical procedure for applying second-order necessary conditions and sufficient conditions. For a system with n state variables, a transition matrix method is used to transform a test for the unboundedness of an $n \times n$ matrix solution of a Riccati equation into a test for a scalar being zero. This allows routine testing of second-order conditions, including the Jacobi no-conjugate-point necessary condition. Four example problems are analyzed: a simple minimum-time problem, the shortest path between two points on a sphere, a multiobjective spacecraft trajectory optimization, and an application of Hamilton's Principle to a circular orbit in an inverse-square gravitational field. In those examples for which second-order conditions are violated and an analytical solution does not exist, a genetic algorithm is used to determine a near-optimal solution.

Introduction

CALCULUS of Variations and Optimal Control Theory have played very important roles in solving optimization problems defined by a performance (cost) functional, differential, and algebraic constraint equations, and associated boundary conditions. Requiring the first variation of the performance functional to vanish leads to the well-known first-order necessary conditions for an optimal solution.[1−4]

These necessary conditions allow one to identify candidates for optimality, called stationary or extremal solutions in order to distinguish them from solutions that have been proven to be optimal. To determine if a stationary solution is indeed optimal, one can formulate sufficient conditions that, if satisfied, guarantee that the solution is at least locally optimal. But, unlike necessary conditions, failing a sufficient coondition test is inconclusive, and there may exist a solution that is optimal even though it does not satisfy the sufficient conditions.

This article develops a procedure for testing second-order necessary conditions and sufficient conditions and applies them to several example optimal control problems. It is a summary of more extensive results and examples presented in Refs. 5–7. This procedure is based on Refs. 8 and 9, which present new sufficient conditions for a weak local minimum of the Bolza optimal control problem. The minimum is said to be weak (as opposed to strong) if both the control and state variations (as opposed to only the state variations) are assumed to be small. An example problem that has both a weak and a strong minimum is given in the Appendix.

Wood's derivation is more complete and straightforward than previous work, and his formulation requires computation of fewer matrix elements. It is also less restrictive than previous theory, e.g., Bryson and Ho[2] (first published in 1969) and the improvements made in Wood and Bryson.[10] However, the sufficient conditions derived include the solution of a matrix Riccati equation be bounded. This is difficult to test numerically because a bounded but rapidly increasing solution can stop the numerical integration and give the false impression that the solution is unbounded. If the solution does become infinite at some point, that point is either the location of a conjugate point or may indicate the presence of a conjugate point elsewhere. An alternative approach by Levin[11] and Prussing[12] is to use a transition matrix to solve the matrix Riccati equation. In the procedure developed in this article, the test for an unbounded matrix is replaced by a test for a (scalar) determinant being zero. The concept of the loss of rank of a matrix as an indication of the existence of a conjugate point appears in other formulations that are based directly on the state-adjoint system.[3,13−16]

First-Order Necessary Conditions

Consider a system described by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] \qquad t_0 \le t \le t_f \tag{1}$$

where $\boldsymbol{x}(t)$ is an n-dimensional state vector and $\boldsymbol{u}(t)$ is a continuous and unconstrained m-dimensional control vector.

Initial conditions are specified by

$$\boldsymbol{x}(t_0) = \boldsymbol{x}^*(t_0) \tag{2}$$

and a $(q+1)$-dimensional terminal constraint vector exists:

$$\boldsymbol{\Psi}[\boldsymbol{x}(t_f), t_f] = 0 \qquad (q \ge 0) \tag{3}$$

Note that a scalar terminal constraint is represented by $q = 0$.

In Refs. 6 and 8 only a scalar terminal constraint ($q = 0$) is considered. In that case the problem is considerably simpler because control variations can be treated as arbitrary without concern for controllability. This is because the scalar terminal constraint either explicitly specifies the final time or implicitly determines it by acting as a "stopping condition."[8] Multiple terminal constraints ($q > 0$) are considered in Refs. 7 and 9 and require a slightly more complicated sufficient condition test. Both cases are discussed in this article.

Consider the problem of minimizing a cost functional of the Bolza form

$$J = \phi[\boldsymbol{x}(t_f), t_f] + \int_{t_0}^{t_f} L[\boldsymbol{x}(t), \boldsymbol{u}(t), t]\,\mathrm{d}t \tag{4}$$

It is convenient to define an augmented terminal function as

$$\Phi[\boldsymbol{x}(t_f), t_f, \nu] = \phi[\boldsymbol{x}(t_f), t_f] + \nu^T \Psi[\boldsymbol{x}(t_f), t_f] \tag{5}$$

where ν is a $(q+1)$-dimensional constant Lagrange multiplier vector. The Hamiltonian function is defined as

$$H[\boldsymbol{x}(t), \boldsymbol{u}(t), \lambda(t), t] = L[\boldsymbol{x}(t), \boldsymbol{u}(t), t] + \lambda^T(t)\boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] \tag{6}$$

where $\lambda(t)$ is an n-dimensional adjoint vector.

The well-known first-order necessary conditions[2] are Eqs. (1–3) and

$$\lambda^{*T}(t) = -H_x[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \lambda^*(t), t] \qquad t_0 \le t \le t_f^* \tag{7}$$

$$\lambda^{*T}(t_f^*) = \Phi_{x(t_f)}\left[\boldsymbol{x}^*(t_f^*), t_f^*, \nu^*\right] \tag{8}$$

$$H_u[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \lambda^*(t), t] = 0^T \qquad t_0 \le t \le t_f^* \tag{9}$$

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where a subscript on a function denotes a partial derivative with respect to that argument and the superscript $*$ denotes the value on the stationary solution.

Equation (9) represents a weak form of the Minimum Principle[1,2] that applies for small variations in an unconstrained control vector (consistent with a weak variation in the cost) and may result in only a local minimum of the Hamiltonian. However, for large allowed variations in the control, the Minimum Principle provides a global minimum of the Hamiltonian and (if the second-order optimality conditions are satisfied) a strong local minimum of the cost.[17] A strong local minimum is not necessarily the global minimum, but in the example problem in the Appendix the strong local minimum cost is unique and is therefore the global minimum cost.

As explained in Refs. 8 and 9, a specified final time is conveniently handled as a terminal constraint in contrast to the treatment in Ref. 2, where it is treated as a control parameter. If the final time is unspecified, one component of the vector ν in Eq. (5), taken to be the last component ν_{q+1} for convenience, can be chosen so that the change in the cost caused by a small change in the final time is equal to zero. This results in the (scalar) necessary condition

$$\Omega\left[\boldsymbol{x}^*(t_f^*), \boldsymbol{u}^*(t_f^*), t_f^*, \nu^*\right] = 0 \tag{10a}$$

where

$$\Omega[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f, \nu] = \frac{\mathrm{d}\Phi}{\mathrm{d}t_f}[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f, \nu] + L[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f] \tag{10b}$$

and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t_f}[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f, \nu] = \Phi_{t_f}[\boldsymbol{x}(t_f), t_f, \nu] + \Phi_{x(t_f)}[\boldsymbol{x}(t_f), t_f, \nu]\boldsymbol{f}[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f] \tag{10c}$$

In addition, one terminal constraint from Eq. (3), taken to be the last component Ψ_{q+1}, can be used to relate a small change in t_f to a change in the state at t_f^*, assuming that a nontangency condition is satisfied given by

$$\left(\frac{\mathrm{d}\Psi_{q+1}}{\mathrm{d}t_f}\right)\left[\boldsymbol{x}(t_f^*), \boldsymbol{u}(t_f^*), t_f^*\right] \neq 0 \tag{11}$$

If necessary, the constraints are renumbered so that the last component satisfies Eq. (11). This results in only q terminal constraints to be considered from the standpoint of controllability.

A solution satisfying the first-order necessary conditions given by Eqs. (1–3) and (7–10) is said to be a stationary solution.

Scalar Terminal Constraint ($q = 0$)

Wood[8] developed sufficient conditions for a weak local minimum of the Bolza problem with a scalar terminal constraint that can be summarized as follows.

Consider the Riccati equation for the $n \times n$ matrix $S(t)$

$$\dot{S}(t) = -S(t)A_1(t) - A_1^T(t)S(t) + S(t)A_2(t)S(t) - A_0(t) \qquad t_0 \le t \le t_f^* \tag{12}$$

where

$$A_0(t) = H_{xx}(t) - H_{xu}(t)H_{uu}^{-1}(t)H_{ux}(t) \tag{13}$$

$$A_1(t) = \boldsymbol{f}_x(t) - \boldsymbol{f}_u(t)H_{uu}^{-1}(t)H_{ux}(t) \tag{14}$$

$$A_2(t) = \boldsymbol{f}_u(t)H_{uu}^{-1}(t)\boldsymbol{f}_u^T(t) \tag{15}$$

with boundary condition

$$S(t_f^*) = S_f\left[\boldsymbol{x}^*(t_f^*), \boldsymbol{u}^*(t_f^*), t_f^*, \nu^*\right] \tag{16}$$

where

$$S_f[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f, \nu] = \Phi_{x(t_f)x(t_f)} - \Omega_{x(t_f)}^T\left(\frac{\mathrm{d}\Psi}{\mathrm{d}t_f}\right)^{-1}\Psi_{x(t_f)} - \Psi_{x(t_f)}^T\left(\frac{\mathrm{d}\Psi}{\mathrm{d}t_f}\right)^{-1}\Omega_{x(t_f)} + \Psi_{x(t_f)}^T\left(\frac{\mathrm{d}\Psi}{\mathrm{d}t_f}\right)^{-1}\left[\Omega_{t_f} + \Omega_{x(t_f)}\boldsymbol{f}\right]\left(\frac{\mathrm{d}\Psi}{\mathrm{d}t_f}\right)^{-1}\Psi_{x(t_f)} \tag{17}$$

Note that if the scalar terminal constraint specifies the final time, only the first term on the right-hand side of Eq. (17) is nonzero.

Sufficient conditions for a minimum cost with scalar terrminal constraint ($q = 0$) can be states as follows:

Theorem 1: Let the function $\boldsymbol{u}^*(t)$, $t_0 \le t \le t_f^*$, be a stationary solution to the optimal control problem with a scalar terminal constraint. If $H_{uu}[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \lambda^*(t), t]$ is positive-definite for $t_0 \le t \le t_f^*$ and if the matrix $S(t)$ defined according to Eqs. (12–17) is finite for $t_0 \le t \le t_f^*$, then the stationary solution furnishes a weak local minimum of the cost.

Second-order necessary conditions are that $H_{uu}(t)$ be positive semidefinite with $S(t)$ finite. These are the classical Legendre–Clebsch and Jacobi necessary conditions in the Calculus of Variations. Unboundedness of the matrix $S(t)$ indicates the existence and location of a conjugate point[17,18] in the case of a scalar terminal constraint.

Multiple Terminal Constraints ($q > 0$)

Wood[9] developed sufficient conditions for a weak local minimum of the Bolza problem with multiple terminal constraints that can be summarized as follows.

Consider a time t_1 such that $t_0 \le t_1 < t_f^*$. Form a Riccati equation for the $n \times n$ matrix $S(t)$ and related matrices $R(t)$ and $Q(t)$ for $t_1 \le t \le t_f^*$.

$$\dot{S}(t) = -S(t)A_1(t) - A_1^T(t)S(t) + S(t)A_2(t)S(t) - A_0(t) \qquad t_1 \le t \le t_f^* \quad (n \times n) \tag{18}$$

$$\dot{R}(t) = \left[S(t)A_2(t) - A_1^T(t)\right]R(t) \qquad t_1 \le t \le t_f^* \quad (n \times q) \tag{19}$$

$$\dot{Q}(t) = R^T(t)A_2(t)R(t) \qquad t_1 \le t \le t_f^* \quad (q \times q) \tag{20}$$

and form a Riccati equation for $\underline{S}(t)$:

$$\dot{\underline{S}}(t) = -\underline{S}(t)A_1(t) - A_1^T(t)\underline{S}(t) + \underline{S}(t)A_2(t)\underline{S}(t) - A_0(t) \qquad t_0 \le t \le t_1 \quad (n \times n) \tag{21}$$

where $A_0(t)$, $A_1(t)$, and $A_2(t)$ are given by Eqs. (13–15).

The boundary conditions are

$$S(t_f^*) = S_f\left[\boldsymbol{x}^*(t_f^*), \boldsymbol{u}^*(t_f^*), t_f^*, \nu^*\right] \tag{22}$$

where

$$S_f[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f, \nu] = \Phi_{x(t_f)x(t_f)} - \Omega_{x(t_f)}^T\left(\frac{\mathrm{d}\Psi_{q+1}}{\mathrm{d}t_f}\right)^{-1}(\Psi_{q+1})_{x(t_f)} - (\Psi_{q+1})_{x(t_f)}^T\left(\frac{\mathrm{d}\Psi_{q+1}}{\mathrm{d}t_f}\right)^{-1}\Omega_{x(t_f)} + (\Psi_{q+1})_{x(t_f)}^T\left(\frac{\mathrm{d}\Psi_{q+1}}{\mathrm{d}t_f}\right)^{-1} \times\left[\Omega_{t_f} + \Omega_{x(t_f)}\boldsymbol{f}\right]\left(\frac{\mathrm{d}\Psi_{q+1}}{\mathrm{d}t_f}\right)^{-1}(\Psi_{q+1})_{x(t_f)} \tag{23}$$

and

$$R(t_f^*) = R_f\left[\boldsymbol{x}^*(t_f^*), \boldsymbol{u}^*(t_f^*), t_f^*\right] \tag{24}$$

where

$$R_f[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f] = \Psi_{x(t_f)}^T - (\Psi_{q+1})_{x(t_f)}^T\left(\frac{\mathrm{d}\Psi_{q+1}}{\mathrm{d}t_f}\right)^{-1}\left(\frac{\mathrm{d}\Psi}{\mathrm{d}t_f}\right)^T \tag{25}$$

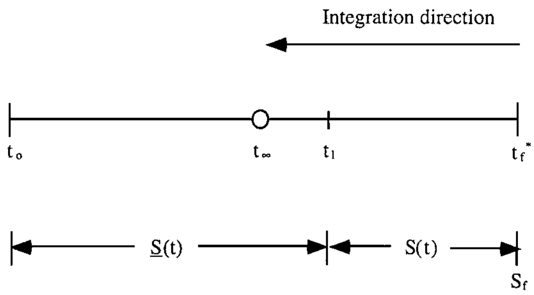

Fig. 1 Multiple terminal constraint terminology.

and

$$Q(t_f^*) = 0_q \tag{26}$$

and the boundary condition for Eq. (21) is

$$\underline{S}(t_1) = S(t_1) - R(t_1)Q^{-1}(t_1)R^T(t_1) \tag{27}$$

Here, Ψ_{q+1} denotes the last component of Ψ, and Ψ denotes the first q components of Ψ. Note that if Ψ_{q+1} explicitly specifies the final time, only the first terms on the right-hand sides of Eqs. (23) and (25) are nonzero.

Sufficient conditions for a minimum cost with multiple terminal constraints ($q > 0$) can be stated as follows:

Theorem 2: Let the function $\boldsymbol{u}^*(t)$, $t_0 \le t \le t_f^*$ be a stationary solution to the optimal control problem with multiple terminal constraints. If $H_{uu}[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \lambda^*(t), t]$ is positive-definite for $t_0 \le t \le t_f^*$ and there exists a time $t_1 < t_f^*$ such that matrix $S(t)$ defined according to Eqs. (18) and (22) is finite for $t_1 \le t \le t_f^*$ and the matrix $\underline{S}(t)$ defined according to Eqs. (18–27) is finite for $t_0 \le t \le t_1$, then the stationary solution furnishes a weak local minimum of the cost.

These sufficient conditions of Wood[9] are less restrictive than those of Bryson and Ho.[2] In Ref. 2 the matrix $\underline{S}(t)$ is required to be finite on the entire interval $t_0 \le t \le t_f^*$. In the less restrictive conditions just cited, the matrix $\underline{S}(t)$ is required to be finite only on the subinterval $t_0 \le t \le t_1 < t_f^*$ for some value of t_1. In fact, if t_1 is initially chosen to be t_0 and the matrix $S(t)$ is finite for $t_0 = t_1 \le t \le t_f^*$, the sufficient condition is satisfied, and the matrix $\underline{S}(t)$ does not need to be calculated. The stationary solution contains no conjugate points.

To summarize, Eq.(18) is solved by backward integration using S_f of Eqs. (22) and (23). If one first chooses t_1 to be t_0 and $S(t)$ is finite on $[t_0, t_f^*]$, then the sufficient condition is satisfied. However, if $S(t)$ is infinite at some point t_∞, another value of t_1 is chosen such that $t_\infty < t_1 < t_f^*$. Equation (21) is then solved using $\underline{S}(t_1)$ from Eq. (27). This is depicted in Fig. 1.

If $\underline{S}(t)$ is finite on $[t_0, t_1]$, then there are no conjugate points on $[t_0, t_f^*]$ because $S(t)$ is finite on $[t_1, t_f^*]$. If there exists more than one value for t_∞, then t_1 must be chosen to be larger than the largest value of t_∞. If $\underline{S}(t)$ is finite, then no conjugate points exist on $[t_0, t_f^*]$.

By contrast with sufficient conditions, second-order necessary conditions are that $H_{uu}(t)$ be positive semidefinite and that no conjugate points exist. These are the classical Legendre–Clebsch and Jacobi necessary conditions in the Calculus of Variations. In the multiple terminal constraint case an unbounded $S(t)$ on $[t_0, t_f^*]$ indicates the possible existence of a conjugate point. A conjugate point actually exists only if $\underline{S}(t)$ is unbounded on $[t_0, t_1]$.

Second-Order Test Procedure

As shown in Refs. 2, 11, and 12, an $n \times n$ matrix Riccati equation can be solved by determining the transition matrix for a $2n$-dimensional linear system:

$$\dot{Z} = PZ \tag{28}$$

where in this application the $2n \times 2n$ coefficient matrix is

$$P(t) = \begin{bmatrix} A_1(t) & -A_2(t) \\ -A_0(t) & -A_1^T(t) \end{bmatrix} \tag{29}$$

One can determine the $2n \times 2n$ transition matrix Φ using $\dot{\Phi} = P\Phi$, where

$$\Phi(t, t_f) = \begin{bmatrix} \Phi_{xx}(t, t_f) & \Phi_{x\lambda}(t, t_f) \\ \Phi_{\lambda x}(t, t_f) & \Phi_{\lambda\lambda}(t, t_f) \end{bmatrix} \tag{30}$$

in terms of the $n \times n$ partitions shown with boundary conditions at the final time $\Phi_{xx}(t_f, t_f) = \Phi_{\lambda\lambda}(t_f, t_f) = I_n$ (identity matrix) and $\Phi_{x\lambda}(t_f, t_f) = \Phi_{\lambda x}(t_f, t_f) = 0_n$.

Using the multiple terminal constraint case as an example, the solution of the matrix Riccati equation can be written in the form

$$S(t) = \Lambda(t)X^{-1}(t) \qquad t_1 \le t \le t_f^* \tag{31}$$

where

$$\Lambda(t) = \Phi_{\lambda x}(t, t_f) + \Phi_{\lambda\lambda}(t, t_f)S_f \tag{32}$$

$$X(t) = \Phi_{xx}(t, t_f) + \Phi_{x\lambda}(t, t_f)S_f \tag{33}$$

Analogous expressions for $\underline{S}(t)$ are

$$\underline{S}(t) = \underline{\Lambda}(t)\underline{X}^{-1}(t) \qquad t_0 \le t \le t_1 \tag{34}$$

where

$$\underline{\Lambda}(t) = \Phi_{\lambda x}(t, t_1) + \Phi_{\lambda\lambda}(t, t_1)\underline{S}(t_1) \tag{35}$$

$$\underline{X}(t) = \Phi_{xx}(t, t_1) + \Phi_{x\lambda}(t, t_1)\underline{S}(t_1) \tag{36}$$

Therefore, if $X(t_c)$ [or $\underline{X}(t_c)$] is singular (determinant equals zero), then $S(t)$ [or $\underline{S}(t)$] is unbounded at time t_c. The test for the unboundedness of the $n \times n$ matrix $S(t)$ [or $\underline{S}(t)$] is replaced by a test for the (scalar) determinant of $X(t)$ [or $\underline{X}(t)$] being zero. This is a simple, practical test. Note that $X(t)$ is actually a factor of the matrix $S^{-1}(t)$, and the test for unboundedness of $S(t)$ has been replaced by an equivalent test for the singularity of $S^{-1}(t)$.

To summarize, the second-order sufficient conditions (Theorem 2) and the procedure for testing for a weak local minimum with multiple terminal constraints are

$$\boldsymbol{x}^*(t) = \boldsymbol{f}[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), t] \tag{37}$$

with $\boldsymbol{x}^*(t_0)$ specified,

$$\Psi\left[\boldsymbol{x}^*(t_f^*), t_f^*\right] = 0 \qquad (q + 1 \text{dimensional}) \tag{38}$$

Euler–Lagrange equation:

$$\lambda^{*T}(t) = -H_x[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \lambda^*(t), t] \qquad t_0 \le t \le t_f^* \tag{39}$$

with $\lambda^{*T}(t_f^*) = \Phi_{x(t_f)}[\boldsymbol{x}^*(t_f^*), t_f^*, \nu^*]$

Optimality condition:

$$H_u[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \lambda^*(t), t] = 0^T \qquad t_0 \le t \le t_f^* \tag{40}$$

Strengthened Legendre–Clebsch condition:

$$H_{uu}[\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \lambda^*(t), t] \text{ is positive-definite} \tag{41}$$

No-conjugate point condition:

$$X(t) = \Phi_{xx}(t, t_f) + \Phi_{x\lambda}(t, t_f)S_f \qquad S_f = S(t_f) \tag{42}$$

Det $[X(t)]$ is nonzero on $t_1 \le t \le t_f^*$ for some $t_1 < t_f^*$ and

$$\underline{X}(t) = \Phi_{xx}(t, t_1) + \Phi_{x\lambda}(t, t_1)(t_1) \tag{43}$$

where $\underline{S}(t_1) = S(t_1) - R(t_1)Q^T(t_1)R^T(t_1)$

Det $[\underline{X}(t)]$ is nonzero on $t_0 \le t \le t_1$

The necessary conditions for a weak local minimum are identical to the preceding with the exception that H_{uu} be only positive semidefinite.

For a scalar terminal constraint ($q = 0$) Theorem 1 applies, which is equivalent to $t_1 = t_0$, and the matrix $\underline{X}(t)$ is not evaluated. The $q = 0$ case requires a one-step process to examine $X(t)$, whereas the $q > 0$ case requires a second step of examining $\underline{X}(t)$ if $X(t)$ is singular because a new value for t_1 must be introduced for which $t_0 < t_1 < t_f^*$.

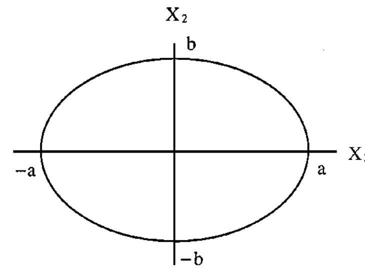

Fig. 2 Elliptic terminal constraint.

Example Problems

Example 1: Minimum Time to an Elliptic Terminal Constraint ($n = 2$ and $q = 0$)

Consider the simple minimum time problem for a vehicle traveling at constant speed $V_0(=1)$ in the x_1–x_2 plane. Expressed in terms of the $n = 2$ state vector $\boldsymbol{x}^T = [x_1 \ \ x_2]$, the equation of motion is $\dot{\boldsymbol{x}} = V_0\sigma$, where σ is a unit-heading vector defined by $\sigma^T = [\cos\theta \ \ \sin\theta]$. The initial time is $t_0 = 0$, and the initial position is $\boldsymbol{x}(0) = 0$. The terminal constraint is the ellipse shown in Fig. 2:

$$\Psi = \frac{1}{2}\left(\frac{x_1^2(t_f)}{a^2} + \frac{x_2^2(t_f)}{b^2} - 1\right) = 0 \qquad \text{where} \qquad a > b$$

This is actually a minimum distance problem recast as a dynamic system. The first-order necessary conditions provide constant-heading stationary solutions having final times $t_f^* = b$ with $\boldsymbol{x}^T(t_f^*) = [0, \pm b]$ and $t_f^* = a$ with $\boldsymbol{x}^T(t_f^*) = [\pm a, 0]$. By inspection of Fig. 2, it is evident that the optimal solution is $t_f^* = b$ with $\boldsymbol{x}^T(t_f^*) = [0, \pm b]$, but it is informative to apply second-order conditions to both stationary solutions.

First stationary solution is $t_f^* = b$ with $\boldsymbol{x}^T(t_f^*) = [0, b]$, $\theta = \pi/2$. From Eqs. (1–6)

$$J = t_f, \qquad (\phi = t_f, L = 0) \tag{44}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f} = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \tag{45}$$

$$H = \lambda_1\cos\theta + \lambda_2\sin\theta \tag{46}$$

$$\Phi = t_f + \nu\left\{\frac{1}{2}\left[\frac{x_1^2(t_f)}{a^2} + \frac{x_2^2(t_f)}{b^2} - 1\right]\right\} \tag{47}$$

Through Eqs. (7–11) one determines $\nu = -b$, $\lambda_1 = 0$, $\lambda_2 = -1$, and $H_{uu} = -\lambda_2 \sin\theta = \sin\theta = 1$. Thus H_{uu} is positive, and the strengthened Legendre–Clebsch condition (41) is satisfied. One must next test for the existence of a conjugate point.

The finiteness of $S(t)$ is determined using the matrix $X(t)$ of Eq. (33). The matrix P of Eq. (29) is calculated to be

$$P = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{48}$$

and the corresponding state transition matrix (30) is determined to be

$$\Phi(t - t_f) = \Phi(t - b) = \begin{bmatrix} 1 & 0 & b-t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{49}$$

The value of S_f is calculated using Eq. (17), and Eq. (33)

$$X(t) = \Phi_{xx} + \Phi_{x\lambda}S_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} b-t & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -b/a^2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - b^2/a^2 + bt/a^2 & 0 \\ 0 & 1 \end{bmatrix} \tag{50}$$

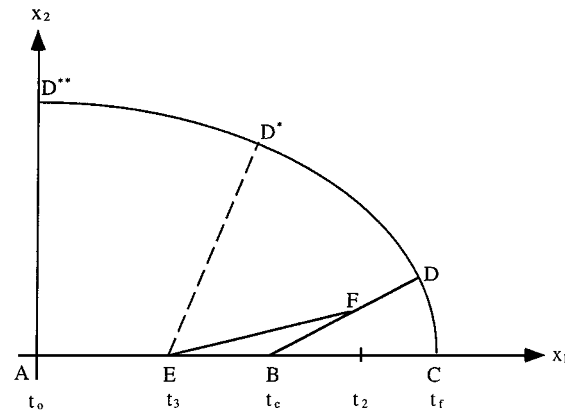

Fig. 3 Conjugate point interpretation for Example 1.

$\text{Det}[X(t)] = 1 - (b/a)^2 + bt/a^2$ with $a > b$. The sign of $\text{Det}[X(t)] = 1 - (b/a)^2$ is positive at $t = 0$ and increases linearly to $\text{Det}[X(t)] = 1$ at $t_f^* = b$. So $\text{Det}[X(t)]$ always has the same sign and is nonzero on $0 \le t \le t_f^* = b$.

Therefore, the no-conjugate point condition is satisfied, and the stationary solution $t_f^* = b$ at $[0, b]$, $\theta = \pi/2$ satisfies the second-order necessary conditions and sufficient conditions. The solution for $[0, -b]$, $\theta = -\pi/2$ yields identical results.

Second stationary solution is $t_f^* = a$ with $\boldsymbol{x}^T(t_f^*) = [a, 0]$, $\theta = 0$.

Using a similar procedure as just stated, one determines $\nu = -a$, $\lambda_1 = -1, \lambda_2 = 0$, and $H_{uu} = -\lambda_1\cos\theta = \cos\theta = 1$; H_{uu} positive and therefore the strengthened Legendre–Clebsch condition (41) is satisfied.

The X matrix is determined to be

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 - a^2/b^2 + at/b^2 \end{bmatrix} \tag{51}$$

$\text{Det}[X(t)] = 1 - (a/b)^2 + at/b^2$ with $a > b$. The sign of $\text{Det}[X(t)] = 1 - (a/b)^2$ is negative at $t = 0$ and increases linearly to $\text{Det}[X(t)] = 1$ at $t_f^* = a$. So $\text{Det}[X(t)]$ is zero at a time t_c for which $0 < t_c < t_f^* = a$, which means that the stationary solution contains a conjugate point and is therefore nonoptimal.

The specific location of the conjugate point is $t_c = (a^2 - b^2)/a$. If the initial point were chosen at $t_2 > t_c$ in Fig. 3, then the stationary solution $t_f^* = a$ at $[a, 0]$, $\theta = 0$ is an optimal path because it does not contain a conjugate point. But, if the initial point is chosen at $t_3 < t_c$, then that stationary solution is no longer the optimal path because it contains a conjugate point at t_c.

The conjugate point in this problem has an interesting geometrical interpretation. It is not the focus of the ellipse, but is always closer to the ellipse center than the focus. The conjugate point is the center of curvature of the ellipse where it crosses the major (x_1) axis, which means that the osculatory circle to the ellipse has its center at the conjugate point $x_1 = t_c$.

This osculatory circle matches both the slope and the curvature of the ellipse at $x_1 = t_f^* = a$. It is the latter point that is significant because all circles centered on the x_1 axis match the slope. The fact that the curvature is also matched means that there are neighboring straight lines from the conjugate point to points on the ellipse near $x_1 = a$ with $x_2 \neq 0$ that have the same length (cost) as the line from $x_1 = t_c$ to $x_1 = a$, $x_2 = 0$. This illustrates the property of same-cost neighboring extremal paths from the conjugate point to the terminal constraint discussed in Refs. 17 and 18, an important property of a conjugate point.

This problem is an excellent example of the reason for the (Jacobi) no-conjugate-point necessary condition. In Fig. 3 paths B-C and B-D have the same length because the conjugate point at B is the center of curvature of the elliptic arc at point C. Thus paths A-E-B-C and A-E-B-F-D have the same length. Assume path A-E-B-C has the minimum distance from point A to the ellipse. But straight-line segment E-F is shorter than the path E-B-F. Thus path A-E-F-D is shorter than path A-E-B-F-D, whose length is the same as A-E-B-C. Thus path A-E-B-C, containing the conjugate point at B, cannot be the minimum distance path from point A to the ellipse.

The minimum distance from point E to point D is obtained by moving point F to point D. The minimum distance from point E to the ellipse is obtained by moving point D to D*, where the path E-D* intersects the ellipse orthogonally. The minimum distance path from point A to the ellipse is obtained by moving point E to point A and point D* to D** for which $t_f^* = b$.

Example 2: Shortest Path Between Two Points on a Sphere ($n = q = 1$)

The minimum distance path between two points on a sphere is a classical demonstration of a conjugate point. In this example, there are two terminal constraints: specified values of ϕ_f and θ_f (ϕ is longitude, θ is latitude), i.e., the location of the final point. The initial point is taken to be $\phi = \theta = 0$.

To find the shortest path between two points, one chooses a coordinate system with the final point specified by $\phi = \phi_f^*$ and $\theta = 0$. As stated in Ref. 2, the problem is to determine $u(\phi)$ to minimize

$$J = \frac{1}{2}\int_0^{\phi f} (u^2 + \cos^2\theta)^{\frac{1}{2}}\, d\phi \tag{52}$$

with $d\theta/d\phi = u$, i.e., the control variable is the slope of the path. From Eq. (6)

$$H = \tfrac{1}{2}(u^2 + \cos^2\theta)^{\frac{1}{2}} + \lambda u \tag{53}$$

There are two terminal constraints:

$$\Psi_1 = \theta(\phi_f) = 0, \qquad \Psi_2 = \phi_f - \phi_f^* = 0 \tag{54}$$

and the stationary solution is easily determined to be $\theta = \phi = 0$.

For this solution $H_{uu} = 1$ and the strengthened Legendre–Clebsch condition (41) is satisfied. One must next test for conjugate points. Because the last component of the terminal constraint $\psi_{q+1} = \psi_2$ specifies the terminal value of the independent variable, only the first term on the right-hand-side of Eq. (23) is nonzero, and the boundary condition is calculated to be $S_f = \Phi_{x(t_f)x(t_f)} = 0$. Because $f_x = 0$, $f_u = 1$, and $H_{\theta u} = 0$, Eqs. (13–15) determine $A_0 = -1$, $A_1 = 0$, and $A_2 = 1$ and from Eq. (29)

$$P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{55}$$

The transition matrix is determined to be

$$\Phi(\phi - \phi_f^*) = \begin{bmatrix} \cos(\phi - \phi_f^*) & -\sin(\phi - \phi_f^*) \\ \sin(\phi - \phi_f^*) & \cos(\phi - \phi_f^*) \end{bmatrix} \tag{56}$$

Because $S_f = 0$ for this example, Eq. (33) simplifies to

$$X(\phi) = \Phi_{xx}(\phi - \phi_f^*) = \cos(\phi - \phi_f^*) \tag{57}$$

For an arbitrary ϕ_f^* a singular value of X occurs at $\phi_\infty = \phi_f^* - \pi/2$. Thus, for $\phi_f^* < \pi/2$ no singular point exists in the interval $0 \le \phi \le \phi_f^*$, and a conjugate point does not exist. However, if $\phi_f^* \ge \pi/2$, a singular point does exist, but because there is more than one terminal constraint, this does not automatically indicate the existence of a conjugate point. One must introduce the intermediate value ϕ_1 separating $S(\phi)$ and $\underline{S}(\phi)$ [and also $X(\phi)$ and $\underline{X}(\phi)$]. The value of ϕ_1 must be chosen to be larger than $\phi_\infty = \phi_f^* - \pi/2$ to avoid the singularity in the interval on which $X(\phi)$ is defined. This is illustrated in Fig. 4.

The analytical solution to Eqs. (18–20) with boundary conditions determined by Eqs. (22–26) to be $S(\phi_f^*) = 0$, $R(\phi_f^*) = 1$, and $Q(\phi_f^*) = 0$ is

$$S(\phi) = \tan(\phi - \phi_f^*) \tag{58a}$$

$$R(\phi) = \sec(\phi - \phi_f^*) \tag{58b}$$

$$Q(\phi) = \tan(\phi - \phi_f^*) \tag{58c}$$

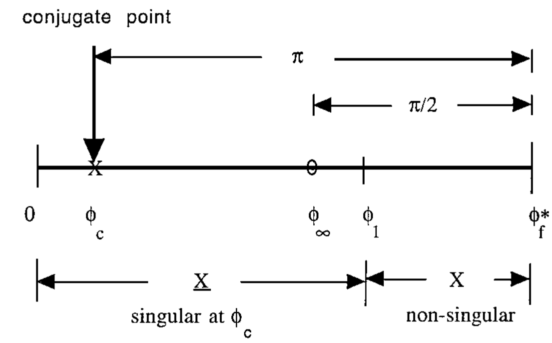

Fig. 4 Conjugate point for Example 2.

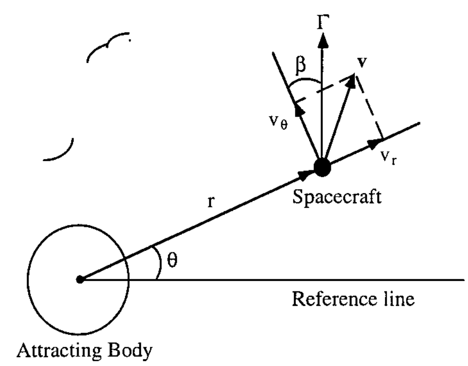

Fig. 5 Variables for spacecraft trajectory.

From Eq. (27)

$$\underline{S}(\phi_1) = S(\phi_1) - R(\phi_1)Q^{-1}(\phi_1)R^T(\phi_1) = -\cot(\phi_1 - \phi_f^*) \tag{59}$$

Then $\underline{X}(\phi)$ is determined to be

$$\underline{X}(\phi) = \Phi_{xx}(\phi, \phi_1) + \Phi_{x\lambda}(\phi, \phi_1)\underline{S}(\phi_1) = \frac{\sin(\phi - \phi_f^*)}{\sin(\phi_1 - \phi_f^*)} \tag{60}$$

By setting Det $[\underline{X}(\phi)] = 0$, one obtains $\sin(\phi_c - \phi_f^*) = 0$, which has a solution $\phi_c = \phi_f^* - \pi$. Thus a conjugate point is located diametrically opposite the terminal point on the sphere. Therefore, for $\phi_f^* < \pi$ no conjugate point exists, as shown in Fig. 4. For $\phi_f^* = \pi$ there exist neighboring extremal paths of the same length (equal to half the circumference of the sphere). And for $\phi_f^* > \pi$ the $u = \theta = 0$ stationary path is not minimum distance because it contains a conjugate point, and there exists a neighboring path of smaller distance.

Note that the existence of a conjugate point and its location is independent of the specific value of ϕ_1.

As shown in Ref. 2 for $\phi_f^* > \pi$, a path can be found that results in a negative value of the second variation of the cost (and therefore a lower cost than the stationary path). It is $\theta(\phi) = A\ \sin[(\pi\phi)/\phi_f^*]$, which yields the negative value of the second variation:

$$\delta^2 J = -(A^2/4\phi_f^*)(\phi_f^{*2} - \pi^2) < 0 \tag{61}$$

Example 3: Optimal Power-Limited Spacecraft Trajectory ($n = 3$, $q = 0$)

Consider a two-dimensional orbit transfer problem with a time-varying thrust provided by a power-limited rocket engine. The spacecraft position is described by polar coordinates with origin at the center of the attracting body as shown in Fig. 5. The thrust angle β (measured relative to the local horizontal) and the magnitude of thrust acceleration Γ are the control variables. This example has a scalar terminal constraint, namely, the specified final time.

The equations of motion are

$$\ddot{r} - r\dot{\theta}^2 = -\mu/r^2 + \Gamma_r \tag{62a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \Gamma_\theta \tag{62b}$$

where Γ_r and Γ_θ are the radial and tangential components of thrust acceleration.

In terms of the ($n = 3$) state variables $\boldsymbol{x}^T = [r \quad v_r \quad v_\theta]$, equations of motion are

$$\dot{r} = v_r \tag{63a}$$

$$\dot{v}_r = v_\theta^2/r - \mu/r^2 + \Gamma \sin\beta \tag{63b}$$

$$\dot{v}_\theta = -v_r v_\theta/r + \Gamma\cos\beta \tag{63c}$$

The value of the polar angle can be computed using $\dot{\theta} = v_\theta/r$, but θ is not needed as a state variable.

In this example the cost functional is defined to be a multiobjective performance index that depends on the total energy at a specified final time t_f and the amount of fuel consumed. The objective is to increase the total energy with a penalty on fuel consumption. A cost is defined that is a linear combination of the negative of the total energy at the final time t_f and the fuel consumed:

$$J = -E(t_f) + \kappa\int_{t_0}^{t_f} \frac{\Gamma^2(t)}{2}\,\mathrm{d}t \tag{64}$$

In Eq. (64) κ is a positive weighting factor for the integral term, which is a measure of fuel consumption for a power-limited rocket engine.[19] Note that the equations of motion must be augmented to include a computation of fuel consumption.

To calculate the amount of fuel consumed, a new variable α is defined by

$$\dot{\alpha} = \Gamma^2/2 \tag{65}$$

This allows the fuel term in Eq. (64) to be computed by integrating the differential equation (65) along with the state equations (63a–63c). Note that α is not a state variable; it is merely a convenient way to evaluate the fuel consumed. The variable α is actually the power-to-mass (PTM) ratio[19] of the engine:

$$\alpha(t) = P_{\max}/m(t) \tag{66}$$

For this system, the adjoint variables corresponding to the state variables are $\lambda^T = [\lambda_r, \lambda_{v_r}, \lambda_{v_\theta}]$, and the Hamiltonian (6) becomes

$$H = \kappa\Gamma^2/2 + \lambda_r v_r + \lambda_{v_r}\left[v_\theta^2/r - \mu/r^2 + \Gamma\sin\beta\right]$$
$$+ \lambda_{v_\theta}[-v_r v_\theta/r + \Gamma\cos\beta] \tag{67}$$

The optimality condition (9)

$$H_u = [H_\Gamma \quad H_\beta] = 0^T \tag{68}$$

yields two equations:

$$\kappa\Gamma + \lambda_{v_r}\sin\beta + \lambda_{v_\theta}\cos\beta = 0 \tag{69a}$$

$$\lambda_{v_r}\Gamma\cos\beta - \lambda_{v_\theta}\Gamma\sin\beta = 0 \tag{69b}$$

Equation (69b) yields the thrust angle for the stationary solution:

$$\tan\beta = \lambda_{v_r}/\lambda_{v_\theta} \tag{70}$$

Equation (69a) then provides the thrust acceleration magnitude for the stationary solution as

$$\Gamma = \sqrt{\lambda_{v_r}^2 + \lambda_{v_\theta}^2}/\kappa \tag{71}$$

For the numerical solution canonical units are used. These are a normalized system of units based on a reference circular orbit. The radius of the reference circular orbit is defined to be one distance unit (DU). If one defines a time unit (TU) such that the reference orbit period is 2πTU, then the value of the gravitational parameter μ has the convenient numerical value of 1 $\mathrm{DU}^3/\mathrm{TU}^2$.

To determine a value for the final energy and an initial state for comparison with other solutions, a final state is arbitrarily chosen to be a circular orbit of radius of 10 DU. The final state vector $\boldsymbol{x}^T = [r \quad v_r \quad v_\theta]$ is then

$$\boldsymbol{x}(t_f)^T = \left[10 \quad 0 \quad 1/\sqrt{10}\right] \tag{72}$$

and the value of the final energy $E_f = -0.05$.

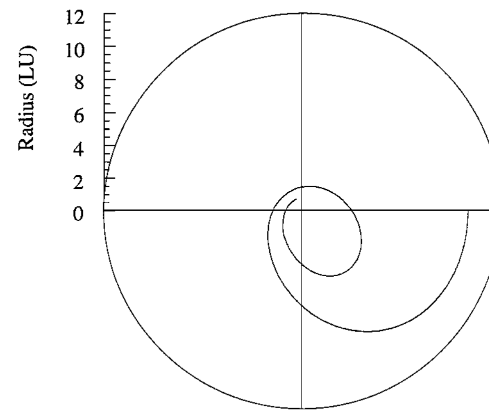

Fig. 6 Stationary power-limited trajectory.

Because the only terminal constraint is the specified final time, Eq. (5) becomes $\Phi[x(t_f), t_f] = -E_f$, and the boundary conditions for the adjoint variables given by Eq. (8) are

$$\lambda^T(t_f) = \left[-1/r^2(t_f) \quad -v_r(t_f) \quad -v_\theta(t_f)\right] \tag{73a}$$

$$= [-0.01 \quad 0 \quad -0.3162] \tag{73b}$$

One can then determine the stationary solution by numerically integrating the equations of motion (63a–63c) backward from the final time to the initial time using the final state [Eq. (72)], the thrust acceleration angle and magnitude [Eqs. (70) and (71)], and the remaining necessary condition (7) with boundary condition (73b). The specified value of the final time is taken to be $t_f^* = 95$ TU, and the weighting factor for the fuel consumption term in the cost [Eq. (64)] is chosen to be $\kappa = 100$. This value of the weighting factor gives approximately equal weight to the energy and fuel terms in the cost. The resulting stationary solution for the trajectory is shown in Fig. 6, with the motion occuring in a counterclockwise direction forward in time.

The numerical integration of the trajectory and the tests for the second-order conditions were performed using the package ODE.[20] This package uses a modified divided difference form of the Adams predictor-corrector integration formulas with local extrapolation. It adjusts the order of the integrator and the step size to satisfy the specified local error. The integration was performed on an IBM RS/6000 J30 with absolute and relative errors of 10^{-7}.

Backward integration of the state and adjoint equations from the final conditions generates a stationary solution with the initial state (Fig. 6) and the amount of fuel consumed as follows:

$$r(0) = 0.8209\ \mathrm{DU} \tag{74a}$$

$$\theta(0) = -10.4620\ \mathrm{rad} \tag{74b}$$

$$v_r(0) = 0.0210\ \mathrm{DU/TU} \tag{74c}$$

$$v_\theta(0) = 1.4142\ \mathrm{DU/TU} \tag{74d}$$

$$\alpha(0) = -0.000732 \tag{74e}$$

The inital value $\alpha(0)$ shown in Eq. (74e) corresponds to a starting value $\alpha(95) = 0$ for the backward numerical integration. This is equivalent to an initial value $\alpha(0) = 0$ and a final value $\alpha(95) = 0.000732$ as a measure of fuel consumed.

The final value of the energy is determined from Eq. (72) to be $E(95) = -0.05$, and the cost [Eq. (64)] is computed to be

$$J = -E(95) + \kappa\alpha(95) = 0.05 + 100\times 0.000732 = 0.1232 \tag{75}$$

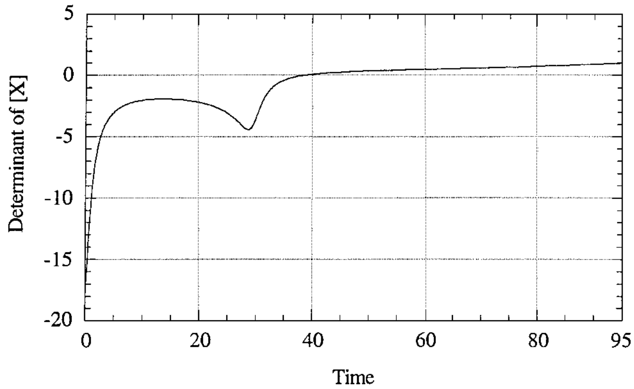

Fig. 7 Determinant of $X(t)$ on the stationary trajectory.

The second-order conditions conditions applied to this trajectory yield

$$H_{uu} = \begin{bmatrix} H_{\Gamma\Gamma} & H_{\Gamma\beta} \\ H_{\beta\Gamma} & H_{\beta\beta} \end{bmatrix}$$

which becomes, using Eqs. (67, 69a and 69b);

$$= \begin{bmatrix} \kappa & 0 \\ 0 & \kappa\Gamma^2 \end{bmatrix} \tag{76}$$

which is positive definite and the strengthened Legendre–Clebsch condition (41) is satisfied.

The (3×3) matrix $X(t)$ is computed by backward numerical integration along with the state and adjoint equations, and the result for the determinant of $X(t)$ is shown in Fig. 7 for $0 \leq t \leq 95$. Note that $\mathrm{Det}[X(t)] = 0$ at $t = 38.6$ TU. Because there is a scalar terminal constraint, this means that the stationary solution contains a conjugate point and is nonoptimal, which implies that for the same initial state that resulted from the backward numerical integration

$$x^T(0) = [0.8209 \quad 0.0210 \quad 1.4142] \tag{77}$$

there exists a solution in a weak, local neighborhood of lower cost for the same specified final time. Note that for this trajectory all of the first-order necessary conditions for an optimal solution are satisfied, but the trajectory is determined to be nonoptimal.

Near-Optimal Trajectory Obtained by a Genetic Algorithm

A genetic algorithm (GA)[21,22] is used to obtain a near-optimal trajectory for the same initial state and final time. The term *near optimal* is used because a GA is well-suited to determine approximate locations in its search space where low-cost solutions exist, but it does not typically yield a highly accurate solution. Details on the particular GA used and its application to this problem are given in Refs. 5–7. The fitness function that the GA seeks to maximize is the reciprocal of the cost J in Eq. (64).

The GA reached the cost of the stationary solution (0.1232) at the ninth generation. As shown in Table 1, the results obtained after 100 generations provide a cost significantly lower (0.1057) than the stationary solution. The GA trajectory has higher final orbit energy (-0.0294) than the stationary solution (-0.05), but the GA solution uses a little more fuel to achieve the higher final energy.

The near-optimal trajectory obtained by the simple GA is shown in Fig. 8, with the motion occuring in a counterclockwise direction forward in time. When compared with the nonoptimal stationary solution shown in Fig. 6 for the same initial state (near the origin) and the same final time, the GA solution is seen to be a tighter-wound spiral with very different final state. Thus the GA has, because of its large search space, determined a solution not in the weak, local neighborhood of the nonoptimal stationary solution, but one that is quite different. So there apparently exists more than one local optimal solution, and the GA has located the lowest cost solution it can find.

The GA solution has a higher final kinetic energy and a smaller final radius, indicating that the optimal strategy is apparently to constrain the potential energy from growing while increasing the kinetic energy. This is qualitatively similar to the maximum final energy trajectory with constant thrust acceleration described in Ref. 23.

Table 1 Comparisons between the stationary and GA trajectories

Variable	Stationary trajectory	GA trajectory
Final energy	0.0500	0.0294
Weighted amount of fuel	0.0732	0.0763
Cost	0.1232	0.1057

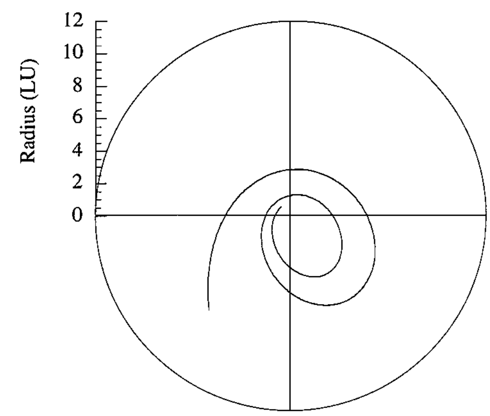

Fig. 8 Power-limited trajectory determined by the GA.

Example 4: Hamilton's Principle for a Circular Orbit ($n = q = 2$)

In classical mechanics Hamilton's Principle states that the natural motion of a conservative dynamical system occurs in such a way that the action has a stationary value.[24] The action is the time integral of the system Lagrangian L, defined as the difference between kinetic and potential energies T − U. To determine whether the stationary value of the action is a minimum, one can apply the second-order conditions considered in this article. The example problem analyzed is a circular orbit in an inverse-square gravitational field, using polar coordinates and canonical units.

First, one determines a path that renders the action integral stationary while moving a spacecraft from $(r_0, \theta_0) = (1, 0 \text{ deg})$ to $(r_f, \theta_f) = (1, 538.6 \text{ deg})$ for a specified final time $t_f = 9.4$ TU. These conditions are satisfied by a circular orbit for an elapsed time corresponding to slightly less than 1.5 orbital periods.

Considering the control variables to be the velocity components, the equations $\dot{\boldsymbol{x}} = \boldsymbol{f}$ describing the circular orbit of $r = 1$ are

$$\dot{r} = u_1 \qquad \text{where} \qquad u_1 = v_r = 0 \tag{78a}$$

$$\dot{\theta} = u_2^2/r \qquad \text{where} \qquad u_2 = v_\theta = 1 \tag{78b}$$

For this problem the action integral is

$$J = \int_0^{t_f} L \,\mathrm{d}t = \int_0^{t_f} \left(\frac{u_1^2 + u_2^2}{2} + \frac{1}{r} \right) \mathrm{d}t \tag{79}$$

Note that this example problem has three terminal constraints ($q = 2$). The final position (r_f, θ_f) and final time t_f are specified:

$$\psi = \begin{bmatrix} r_f - r_f^* \\ \theta_f - \theta_f^* \\ t_f - t_f^* \end{bmatrix} \tag{80}$$

As before, it is convenient to define the last component to specify the final time.

For this system the adjoint variables are $\lambda^T = [\lambda_r, \lambda_\theta]$, and the Hamiltonian function is

$$H = \left[\left(u_1^2 + u_2^2\right)/2 \right] + 1/r + \lambda_r u_1 + \lambda_\theta (u_2/r) \tag{81}$$

From the optimality conditions (9), (78a, 78b), and (81)

$$\frac{\partial H}{\partial u_1} = u_1 + \lambda_r = 0 \Rightarrow \lambda_r = -u_1 = 0 \tag{82a}$$

$$\frac{\partial H}{\partial u_2} = u_2 + \lambda_\theta = 0 \Rightarrow \lambda_\theta = -u_2 = -1 \tag{82b}$$

and $H_{uu} = I_2 > 0$. Thus, the strengthened Legendre–Clebsch condition (41) is satisfied.

The values of A_0, A_1, and A_2 are determined by Eqs. (13–15) as

$$A_0 = \begin{bmatrix} 2(1+\lambda_\theta u_2)/r^3 - \lambda_\theta^2/r^4 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{83a}$$

$$A_1 = \begin{bmatrix} 0 & 0 \\ -u_2/r^2 + \lambda_\theta/r^3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} \tag{83b}$$

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1/r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{83c}$$

Finally the matrix in Eq. (29) is obtained as

$$P = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{84}$$

The matrices Φ_{xx} and $\Phi_{x\lambda}$ are determined to be

$$\Phi_{xx} = \begin{bmatrix} \cos(t-t_f) & 0 \\ -2\sin(t-t_f) & 1 \end{bmatrix} \tag{85a}$$

$$\Phi_{x\lambda} = \begin{bmatrix} -\sin(t-t_f) & -2+2\cos(t-t_f) \\ 2-2\cos(t-t_f) & 3(t-t_f) - 4\sin(t-t_f) \end{bmatrix} \tag{85b}$$

Because the specified final time is taken to be the last component of the constraint vector (80), as in Examples (2) and (3), only the first term on the right-hand side of Eq. (23) is nonzero. As in Example (2), one initially chooses the value of t_1 to be t_0.

The matrix $X(t)$ is determined from $X = \Phi_{xx} + \Phi_{x\lambda} S_f$ with $S_f = 0_2$ to be

$$X(t) = \begin{bmatrix} \cos(t-t_f) & 0 \\ -2\sin(t-t_f) & 1 \end{bmatrix} \tag{86}$$

$\mathrm{Det}[X(t)] = \cos(t-t_f)$ and the value of t_∞ is obtained by setting $\cos(t_\infty - t_f) = 0$ to yield $t_\infty = t_f - \pi/2$. For $t_f = 9.4$ the value of $t_\infty = 7.8$. Therefore as in the multiple terminal constraint, Example (2), one needs to investigate the matrix $\underline{X}(t)$ by choosing a value of t_1 such that $t_\infty < t_1 < t_f$.

The analytical solution to Eqs. (18–20) with boundary conditions given by $S(t_f) = 0_2$, $R(t_f) = I_2$, and $Q(t_f) = 0_2$ is obtained using Mathematica[25]:

$$S(t) = \begin{bmatrix} \tan(t-t_f) & 0 \\ 0 & 0 \end{bmatrix} \tag{87}$$

$$R(t) = \begin{bmatrix} \sec(t-t_f) & 2\tan(t-t_f) \\ 0 & 1 \end{bmatrix} \tag{88}$$

$$Q(t) = \begin{bmatrix} \tan(t-t_f) & -2+2\sec(t-t_f) \\ -2+2\sec(t-t_f) & 3(t_f-t) + 4\tan(t-t_f) \end{bmatrix} \tag{89}$$

The value of $\underline{S}(t_1)$ is then

$$\underline{S}(t_1) = S(t_1) - R(t_1)Q^{-1}(t_1)R^T(t_1) = \frac{1}{\Delta}\begin{bmatrix} \csc[(t_f-t_1)/2][3t_1\cos(t_f-t_1) + 4\sin(t_f-t_1)]/2 & -2\sin[(t_f-t_1)/2] \\ -2\sin[(t_f-t_1)/2] & \cos(t_f-t_1) \end{bmatrix} \tag{90}$$

where $\Delta = 3t_1\cos[(t_f-t_1)/2] + 8\sin[(t_f-t_1)/2]$. $\mathrm{Det}[\underline{X}(t)]$ is determined from Eq. (90) as

$$\mathrm{Det}[\underline{X}(t)] = \frac{\csc[(t_f-t_1)/2]\sin[(t-t_f)/2]\{-3t\cos[(t-t_f)/2] + 3t_f\cos[(t-t_f)/2] + 8\sin[(t-t_f)/2]\}}{-3t_f\cos[(t_f-t_1)/2] + 3t_1\cos[(t_f-t_1)/2] + 8\sin[(t_f-t_1)/2]} \tag{91}$$

The numerator of $\mathrm{Det}[\underline{X}(t)]$ set equal to zero yields (independent of the value of t_1)

$$\sin\eta = 0 \tag{92}$$

where $\eta = (t - t_f)/2$ and

$$-3t\cos\eta + 3t_f\cos\eta + 8\sin\eta = 0 \tag{93a}$$

which yields

$$\tfrac{3}{4}\eta = \tan\eta \tag{93b}$$

Solutions to Eq. (92) occur at multiples of the orbit period prior to the final time and indicate the locations of conjugate points. To determine solutions to Eq. (93b), note that intersections of a straight line through the origin having slope less than unity with the tangent function occur on every branch of the tangent curve except the first.

The two solutions of Eq. (93b) closest to the final time are determined numerically to be $t_f - t = 2.8135\pi$ (8.8389 rad = 506.43 deg) and 4.8906π (15.3643 rad = 880.31 deg) (Ref. 26). So for $t_f = 9.4$ a conjugate point exists at $t_c = 9.4 - 8.8389 = 0.5611$, and the circular orbit violates the no-conjugate-point condition if the final angle exceeds 506.43 deg. The alternate path, which has a lower cost (value of the action), is determined by a GA and is described in the following section.

The conjugate points in this example have an interesting interpretation. They occur at the same times as the singularities of the matrix $\partial \boldsymbol{r}(t_f)/\partial \boldsymbol{v}(t)$, which is one of the partitions of the (position-velocity) state transition matrix evaluated along the circular orbit.[27] This partition is the matrix that must be inverted to determine a velocity variation $\delta\boldsymbol{v}(t)$ at any time t to satisfy a specified position variation at the final time $\delta\boldsymbol{r}(t_f)$. Nonexistence of this inverse indicates that an arbitrary final position variation cannot be achieved for that time.

Orbit Determined by a GA

When the circular orbit contains a conjugate point, there exists an alternate orbit of lower cost (action). A GA is used to determine this alternate orbit, which must connect the same initial and final positions (but not the velocities) for the same terminal times. Details of the GA and its application are given in Refs. 5–7.

The only difference on the alternate orbit is its initial velocity. The initial velocity components are chosen by the GA, and the resulting orbit is determined by integrating the equations of motion. Simultaneously, the action rate is integrated to determine the value of the action, denoted by $\alpha(t_f) = J$ of Eq. (79).

The integrated equations are

$$\dot r = u_1 \qquad (u_1 = v_r) \tag{94a}$$

$$\dot\theta = u_2/r \qquad (u_2 = v_\theta) \tag{94b}$$

$$\dot v_r = u_2^2/r - \mu/r^2 \tag{94c}$$

$$\dot v_\theta = -(u_1 u_2/r) \tag{94d}$$

$$\dot\alpha = (u_1^2 + u_2^2)/2 + 1/r \tag{94e}$$

The objective of the GA is to choose values for the two initial velocity components, expressed in terms of initial velocity magnitude v_0 and angle β_0. Then the initial state is described by

$$\boldsymbol{x}(t_0) = [r \quad \theta \quad v_r \quad v_\theta]^T = [1 \quad 0 \quad v_0\sin\beta_0 \quad v_0\cos\beta_0]^T \tag{95}$$

Table 2 Comparisons between circular and GA orbits

Variable	Circular orbit	GA trajectory
Cost	14.1	7.5
r_f	1.00	1.06
θ_f	3.12	3.13

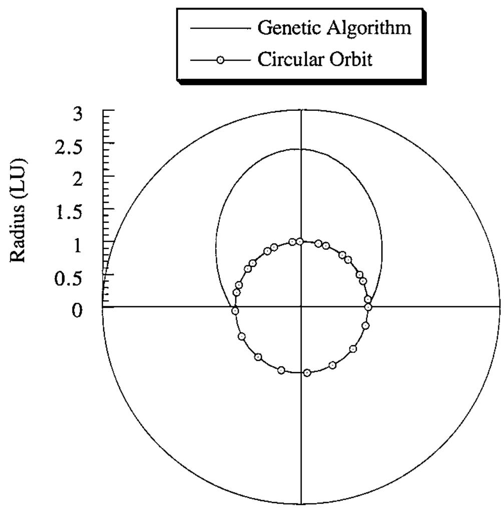

Fig. 9 Orbit determined by the GA.

In contrast to Example 3 where only the final time was specified, the fitness function to be maximized by the GA needs to include penalty functions that force the final position components to be equal to those on the circular orbit. The final position is specified by $r = 1$ DU and $\theta = 3.12$ rad (178.6 deg = 538.6 deg − 360 deg). The cost (to be minimized) is given by the variable J in Eq. (84), and the GA fitness function (to be maximized) is chosen to be

$$\frac{1}{J + [1.0 - r_f(\mathrm{GA})]^2/0.05^2 + [3.12 - \theta_f(\mathrm{GA})]^2/0.1^2} \tag{96}$$

where the weighting factors 0.05 and 0.1 were determined by trial and error. For comparison, the value of the cost (action) for the circular orbit $r = 1$, $u_1 = 0$, and $u_2 = 1$ is calculated from Eq. (79) as $\frac{3}{2}t_f = \frac{3}{2}9.4 = 14.1$.

After the ninth generation GA determines a near-optimal orbit of significantly lower cost than the circular orbit, as shown in Table 2. This GA orbit is the less-than-one-revolution elliptic orbit shown in Fig. 9. Note that the value of the cost (action) on the elliptic orbit is 7.5, considerably less than the value on the circular orbit of 14.1.

The alternate orbit between the same points in space at the same terminal times has an interesting interpretation in terms of Lamberts Problem[27] in orbital mechanics, which is the determination of the orbit that connects two specified points in a specified time. The circular orbit is a multirevolution solution to Lambert's Problem: the transfer time of 9.4 TU is equal to a complete orbit period ($2\pi = 6.28$ TU) plus the time required to travel through another 178.6 deg, which is 3.12 TU. By contrast the GA orbit is a less-than-one-revolution orbit between the same endpoints and final time.

The interpretation of the conjugate point in this example is that it represents the minimum time for which a multirevolution solution exists for the same endpoints and final time as a less-than-one-revolution orbit. This is verified using the procedure outlined in Refs. 28 and 29. The connection between the singularity of the transition matrix partition and the minimum time for a multirevolution orbit was observed by Stern.[30]

Conclusions

An efficient procedure is developed for applying second-order necessary conditions and sufficient conditions for a weak local minimum in the Bolza optimal control problem with scalar or multiple terminal constraints. This procedure tests for the existence of a conjugate point by solving a linear system of equations to determine if a (scalar) determinant is zero. Four example problems are analyzed to illustrate this procedure. Under some conditions solutions that satisfy the first-order necessary conditions for an optimal solution contain a conjugate point and are therefore nonoptimal. Lower cost optimal or near-optimal solutions are then determined either analytically or using a genetic algorithm.

Appendix: Weak Minimum vs Strong Minimum

For the following problem, based on an example in Ref. 31 the stationary solution satisfies the second-order necessary conditions and sufficient conditions for a weak local minimum. However, this problem also has a strong minimum that is the global minimum. The problem, for which $n = q = 1$, is as follows:

Minimize:

$$J = \int_0^{t_f} (u^2 - 1)^2 \, \mathrm{d}t \tag{A1}$$

with

$$\dot{x} = u \qquad x(0) = 0 \qquad x(t_f) = 0.8 \tag{A2}$$

The final time t_f is specified to be equal to 1, resulting in two terminal contraints ($q = 1$). The Hamiltonian function (6) is given by

$$H = (u^2 - 1)^2 + \lambda u \tag{A3}$$

and the augmented terminal function (5) is

$$\Phi = \nu_1[x(t_f) - 0.8] + \nu_2[t_f - 1] \tag{A4}$$

Equations (7) and (8) yield the conditions

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = 0 \qquad \lambda(1) = \frac{\partial \Phi}{\partial x(t_f)} = \nu_1 \tag{A5}$$

From the optimality condition (9)

$$\frac{\partial H}{\partial u} = 4u(u^2 - 1) + \nu_1 = 0 \tag{A6}$$

Using Eq. (A6) and the boundary conditions, a solution is

$$u = \text{const} = 0.8 \Rightarrow x(t) = 0.8t \qquad \nu_1 = 1.15 \tag{A7}$$

To test the second-order conditions,

$$H_{uu} = 4(3u^2 - 1)_{u=0.8} = 3.68 > 0 \tag{A8}$$

and the strengthened Legendre–Clebsch condition (41) is satisfied. To test for a conjugate point,

$$H_{xu} = H_{xx} = f_x = S_f = 0 \qquad f_u = 1$$

$$A_0 = A_1 = 0 \qquad A_2 = 1/3.68$$

and

$$P = \begin{bmatrix} 0 & -1/3.68 \\ 0 & 0 \end{bmatrix} \tag{A9}$$

The transition matrix is

$$\Phi(t - t_f) = \begin{bmatrix} 1 & -(t - t_f)/3.68 \\ 0 & 1 \end{bmatrix} \tag{A10}$$

Choosing $t_1 = 0$,

$$X(t) = \Phi_{xx} + \Phi_{x\lambda} S_f = \Phi_{xx} = 1 \tag{A11}$$

and the solution $u = 0.8$ satisfies the second-order necessary conditions and sufficient conditions for a weak local minimum with a cost [Eq. (A1)] of $J^* = (0.36)^2 = 0.1296$.

However, by inspection of the cost functional, a piecewise continuous control $u = \pm 1$ results in a zero cost. The boundary conditions

$x(0) = 0$ and $x(1) = 0.8$ can be satisfied by an infinite number of equal-cost solutions, one of which is

$$u(t) = +1 \qquad 0 \leq t \leq 0.9 \tag{A12a}$$

$$u(t) = -1 \qquad 0.9 \leq t \leq 1 \tag{A12b}$$

Note that the control is discontinuous and represents a strong variation. The corresponding solution for the state is

$$x(t) = t \qquad 0 \leq t \leq 0.9 \tag{A13a}$$

$$x(t) = 1.8 - t \qquad 0.9 \leq t \leq 1 \tag{A13b}$$

with the cost $J^* = 0$, the (strong) global minimum cost.

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