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Engineering Notes

Second-Order Necessary Conditions and Sufficient Conditions Applied to Continuous-Thrust Trajectories

John E. Prussing* and Suzannah L. Sandrik†
University of Illinois at Urbana–Champaign, Urbana, Illinois 61801

Introduction

In optimal control theory, requiring the first variation of the performance functional to vanish leads to well-known first-order necessary conditions (NC) for an optimal solution.[1] These NC allow one to identify candidates for optimality, called stationary or extremal solutions, to distinguish them from solutions that have been proven to be optimal. To determine if a stationary solution is indeed optimal, one must also test the second-order Jacobi no-conjugate-point NC, which applies if the trajectory is smooth. Also, one can formulate sufficient conditions (SC) that, if satisfied, guarantee that the solution is at least locally optimal.

In this Note, a procedure developed by Jo and Prussing[2–4] for testing second-order NC and SC is streamilned and applied to an example optimal continuous-thrust trajectory with multiple-terminal constraints that yields a different type of result compared to previous examples in Refs. 2, 4, and 5. The procedure is based on earlier work by Wood[5,6] that derives new, less restrictive SC for a weak local minimum of the Bolza optimal control problem. However, those SC require that the solution of a matrix Riccati equation be bounded. This is difficult to test numerically because a bounded but rapidly increasing solution can stop the numerical integration and give the false impression that the solution is unbounded. The procedure described and illustrated in this Note replaces the test for an unbounded matrix[5,6] by a test for a (scalar) determinant being zero.

Problem Formulation

The problem formulation is described here, along with some definitions that are explained more fully in Ref. 2 (which combines Refs. 3 and 4). Consider a system described by

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t], \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{1}$$

for $t_0 \le t \le t_f$, where $\boldsymbol{x}(t)$ is an n-dimensional state vector, $\boldsymbol{u}(t)$ is an unconstrained m-dimensional control vector, and the final time t_f may be specified or free.

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*Professor, Department of Aerospace Engineering, 306 Talbot Laboratory. Fellow AIAA.

†Graduate Student, Department of Aerospace Engineering, 306 Talbot Laboratory. Member AIAA.

A cost functional of the Bolza form is to be minimized,

$$J = \phi[\boldsymbol{x}(t_f), t_f] + \int_{t_0}^{t_f} L[\boldsymbol{x}(t), \boldsymbol{u}(t), t]\,\mathrm{d}t \tag{2}$$

A $(q+1)$-dimensional terminal constraint vector exists,

$$\boldsymbol{\psi}[\boldsymbol{x}(t_f), t_f] = \boldsymbol{0} \tag{3}$$

where $q \ge 0$ and a single (scalar) terminal constraint corresponds to $q = 0$.

For problems having only a single terminal constraint ($q = 0$), the second-order test is simpler because control variations can be treated as arbitrary without concern for controllability. This is because the single terminal constraint either explicitly specifies the final time or implicitly determines it by acting as a stopping condition. For this reason, there is always at least one terminal constraint.

Problems having multiple terminal constraints ($q > 0$) may require a more complicated two-step second-order test. In this Note, the procedures for both single and multiple terminal constraint continuous-thrust trajectories are explained. An illustrative multiple terminal constraint example is presented.

As in Chapter 2 of Ref. 1, it is convenient to define an augmented terminal function as

$$\boldsymbol{\Phi}[\boldsymbol{x}(t_f), t_f, \boldsymbol{\nu}] = \phi[\boldsymbol{x}(t_f), t_f] + \boldsymbol{\nu}^T \boldsymbol{\psi}[\boldsymbol{x}(t_f), t_f] \tag{4}$$

where $\boldsymbol{\nu}$ is a $(q+1)$-dimensional constant Lagrange multiplier vector.

The Hamiltonian function is defined as

$$H[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t), t] = L[\boldsymbol{x}(t), \boldsymbol{u}(t), t] + \boldsymbol{\lambda}^T(t)\boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), t] \tag{5}$$

where $\boldsymbol{\lambda}(t)$ is an n-dimensional adjoint vector.

An additional function that is needed is defined as

$$\Omega[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f, \boldsymbol{\nu}] = \frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t_f}[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f, \boldsymbol{\nu}] + L[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f] \tag{6}$$

with

$$\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t_f}[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f, \boldsymbol{\nu}] = \boldsymbol{\Phi}_{t_f}[\boldsymbol{x}(t_f), t_f, \boldsymbol{\nu}] + \boldsymbol{\Phi}_{x(t_f)}[\boldsymbol{x}(t_f), t_f, \boldsymbol{\nu}]\boldsymbol{f}[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f] \tag{7}$$

where $\boldsymbol{\Phi}_{t_f}$ and $\boldsymbol{\Phi}_{x(t_f)}$ represent partial derivatives of the function $\boldsymbol{\Phi}$ in Eq. (4). In addition, one terminal constraint from Eq. (3), taken to be the last (or only, if $q = 0$) component ψ_{q+1}, can be used to relate a small change in t_f to a small change in the state at the optimal final time t_f^*, assuming that a nontangency condition is satisfied given by

$$\frac{\mathrm{d}\psi_{q+1}}{\mathrm{d}t_f}[\boldsymbol{x}(t_f), \boldsymbol{u}(t_f), t_f] \ne 0 \tag{8}$$

If necessary, the constraints are renumbered so that the last component satisfies Eq. (8). This results in only q terminal constraints to be considered from the standpoint of controllability.

Second-Order Test Procedure

The second-order test procedure described hereafter (Ref. 2 with improved notation) requires determination of a $2n \times 2n$ transition

4) Simultaneously calculate $\det[X(t)]$, given by Eq. (13), using the matrix S_f from Eq. (14).

The remainder of the procedure depends on whether the terminal constraint is single ($q = 0$) or multiple ($q > 0$). The remaining condition for a single-terminal constraint is denoted by 5, and the remaining conditions for multiple-terminal constraints are denoted by 6 and 7. Q2

5) For $q = 0$, if there exists a time t_c with $t_0 \le t_c \le t_f^*$ for which $\det[X(t_c)] = 0$, then a conjugate point exists at time t_c and the trajectory is nonoptimal. However, if $\det[X(t)]$ is nonzero for $t_0 \le t \le t_f^*$, then the stationary solution furnishes a weak local minimum of the cost. The procedure for a single-terminal constraint is completed at this point.

6) For $q > 0$, initially select the time t_1 to be the initial time t_0. If $\det[X(t)]$ is nonzero for $t_0 \le t \le t_f^*$, then the stationary solution furnishes a weak local minimum of the cost. The procedure for multiple-terminal constraints is completed at this point. However, if there exists a time t_z with $t_0 \le t_z \le t_f^*$ for which $\det[X(t_z)] = 0$, then a conjugate point may exist. Further testing is required as described in step 7.

7) Select a new time t_1 for which $t_z < t_1 < t_f^*$. Then $\det[X(t)]$ will be nonzero for $t_1 \le t \le t_f^*$. Calculate $\det[\hat{X}(t)]$ for $t_0 \le t \le t_1$ using Eq. (15) and other matrices defined by Eqs. (16–20b). If $\det[\hat{X}(t_c)] = 0$ for $t_0 \le t_c \le t_1$, a conjugate point exists at time t_c and the trajectory is nonoptimal. However, if $\det[\hat{X}(t)]$ is nonzero for $t_0 \le t \le t_1$, then the stationary solution furnishes a weak local minimum of the cost. The procedure for multiple-terminal constraints is completed at this point.

Example Problem

Kechichian[9] reformulated an optimal low-thrust solution by Edelbaum[10,11] using optimal control theory. Both formulations are based solely on first-order NC, and so it is of interest to determine whether second-order NC and SC for an optimal solution are satisfied by this solution. There are $n = 2$ state variables, $m = 1$ control variable, and two terminal constraints ($q = 1$), namely, the specified final values of the inclination and circular orbit velocity. In Ref. 10, Edelbaum derives two system equations by averaging over a fast variable (true anomaly):

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{2\Gamma \sin\beta}{\pi V} \tag{21}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\Gamma \cos\beta \tag{22}$$

where i is the orbital inclination, V is the (scalar) orbital velocity, and the thrust acceleration Γ is assumed constant. The angle β is the (out-of-plane) thrust yaw angle.

The minimum-propellant problem is then cast as a minimum-time problem (because Γ is constant) from initial conditions (i_0, V_0) to specified final conditions (i_f, V_f), where V_0 and V_f are the circular orbit velocities in the specified initial and final circular orbits.

The Hamiltonian function for this problem is then (see Ref. 9)

Q3
$$H = 1 + \lambda_i[(2\Gamma/\pi V)\sin\beta] - \lambda_V \Gamma \cos\beta \tag{23}$$

In Ref. 9 it is shown that

$$\lambda_i = \text{const} = -\frac{\pi V \sin\beta}{2\Gamma} \tag{24}$$

$$\lambda_V(t) = \frac{\cos\beta}{\Gamma} \tag{25}$$

The velocity on the stationary solution is

$$V(t) = \left(V_0^2 - 2V_0\Gamma t\cos\beta_0 + \Gamma^2 t^2\right)^{\frac{1}{2}} \tag{26}$$

with

$$\tan\beta_0 = \frac{\sin[(\pi/2)\Delta i_f]}{(V_0/V_f) - \cos[(\pi/2)\Delta i_f]} \tag{27}$$

with $\Delta i_f = |i_0 - i_f|$. Also,

$$\Delta i(t) = \frac{2}{\pi}\left[\tan^{-1}\left(\frac{\Gamma t - V_0\cos\beta_0}{V_0\sin\beta_0}\right) + \frac{\pi}{2} - \beta_0\right] \tag{28}$$

$$\tan\beta(t) = \frac{V_0\sin\beta_0}{(V_0\cos\beta_0 - \Gamma t)} \tag{29}$$

Several numerical examples of stationary solutions for low-Earth-orbit (LEO) to geosynchronous-Earth-orbit (GEO) transfer are determined in Ref. 9 with $r_0 = 7000$ km; various values of i_0; $r_f = 42{,}166$ km; and $i_f = 0$. The value of Γ is assumed to be $3.5 \times 10^{-7}\,\text{km/s}^2$. Two of these numerical examples will be examined: $i_0 = 28.5$ deg, for which the transfer time for the stationary trajectory is 191 days, and $i_0 = 90$ deg, with a transfer time of 335 days.

Applying the second-order procedure, one uses Eqs. (23–25) to determine that

$$H_{\beta\beta} = -\lambda_i(2\Gamma/\pi V)\sin\beta + \lambda_V\Gamma\cos\beta = 1 > 0 \tag{30}$$

satisfying the strengthened Legendre–Clebsch condition in steps 6 and 7. Q4

The matrices in Eqs. (11a–11c) are determined to be

$$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & -\sin^2\beta(2+\cos^2\beta)/V^2 \end{bmatrix} \tag{31a}$$

$$A_1 = \begin{bmatrix} 0 & -2\Gamma\sin\beta(1+\cos^2\beta)/\pi V^2 \\ 0 & -\Gamma\cos\beta\sin^2\beta/V \end{bmatrix} \tag{31b}$$

$$A_2 = \Gamma^2\begin{bmatrix} (2\cos\beta/\pi V)^2 & 2\cos\beta\sin\beta/\pi V \\ 2\cos\beta\sin\beta/\pi V & \sin^2\beta \end{bmatrix} \tag{31c}$$

Because the final time is not specified, calculation of the matrix S_f requires evaluation of all of the terms in Eq. (14). The result is

$$S_f = \frac{\sin^2\beta_f}{\Gamma\cos\beta_f V_f}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{32}$$

where the final thrust angle β_f is determined as shown in Ref. 9 using

$$\Delta V_{\text{tot}} = V_0\cos\beta_0 - \frac{V_0\sin\beta_0}{\tan[(\pi/2)\Delta i_f + \beta_0]} \tag{33}$$

with β_0 given by Eq. (27). The value of β_f is calculated using Eq. (29) by substituting ΔV_{tot} for Γt. Finally, the value of the transfer time is given by

$$t_f = \Delta V_{\text{tot}}/\Gamma \tag{34}$$

The 4×4 matrix P of Eq. (10) can now be assembled. The 4×4 matrix $\Theta(t, t_f)$ is calculated using Eqs. (9a) and (9b), and the determinant of the 2×2 matrix $X(t)$ is calculated using Eq. (13).

Figure 1 shows $\det[X(t)]$ for two numerical examples from Ref. 9. When step 6 is followed and the value of $t_1 = t_0 = 0$ is selected, $\det[X(t)]$ is seen to be nonzero for the entire $i_0 = 28.5$ deg trajectory, but for $i_f = 90$ deg there exists a time $t_z = 245$ days for which $\det[X(t_z)] = 0$. Thus, the 28.5-deg trajectory is optimal (a weak local minimum), and the 90-deg trajectory may contain a conjugate point. To determine whether a conjugate point exist, step 7 must be followed.

The calculation of $\det[\hat{X}(t)]$ for this example requires calculation of the 2×1 matrix $R(t)$, scalar $Q(t)$, and 2×2 matrix $\hat{S}(t_1)$ given in Eqs. (19a), (20a), and (16). The boundary condition R_f in Eq. (19b) is

$$R_f = \begin{bmatrix} 1 \\ \dfrac{2\tan\beta_f}{\pi V_f} \end{bmatrix} \tag{35}$$

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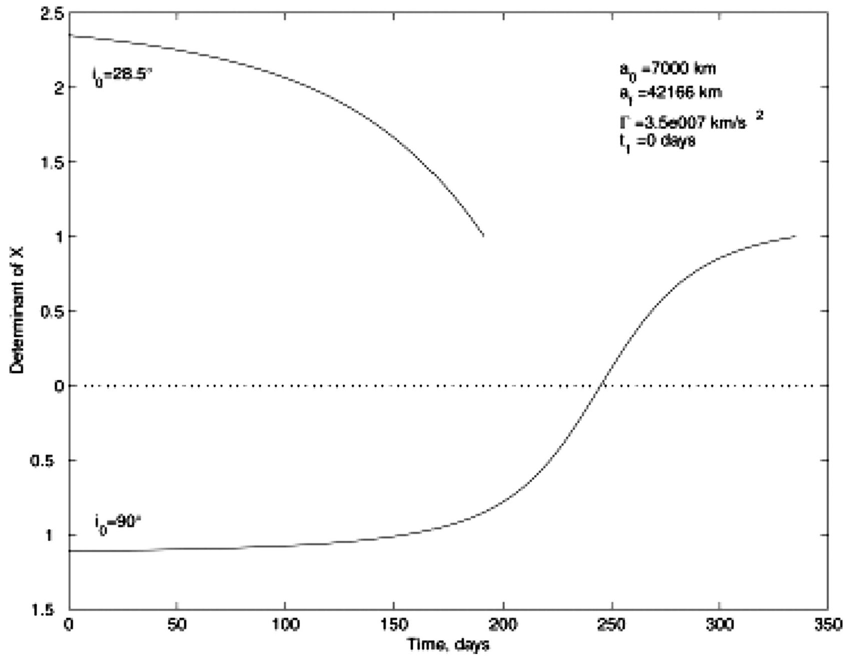

Fig. 1 Determinant of $X(t)$.

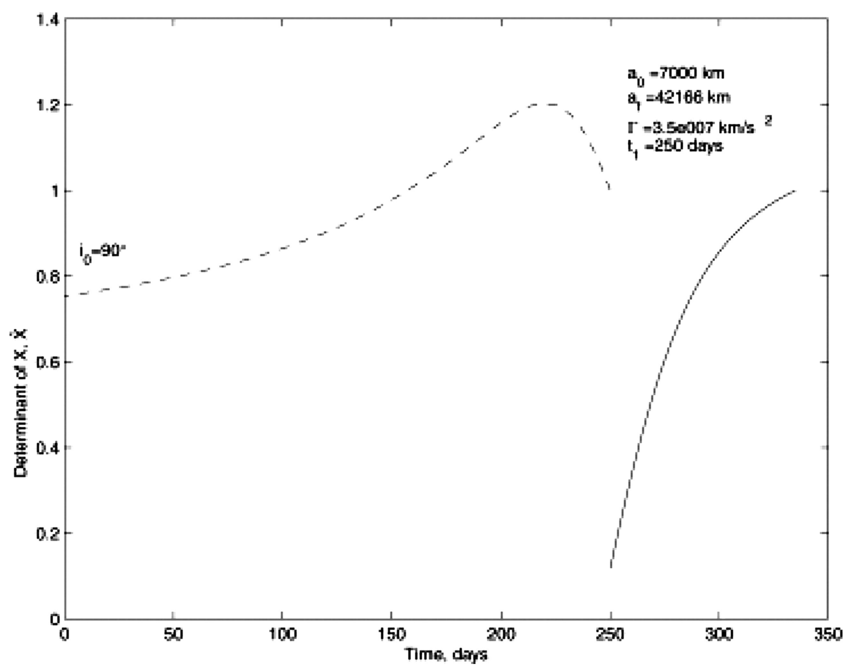

Fig. 2 Determinant of $X(t)$ for $250 \le t \le 335$ and $\hat{X}(t)$ for $0 \le t \le 250$.

A value of $t_1 = 250 > t_z = 245$ days is selected and $\det[\hat{X}(t)]$ is evaluated for $0 \le t \le 250$. As seen in Fig. 2, the value of $\det[\hat{X}(t)]$ is nonzero and no conjugate point exists. Therefore, the 90-deg transfer is also optimal (a weak local minimum), as are all of the other numerical examples tested for LEO to GEO transfers. This
Q5 is in contrast to the multiple terminal constraint examples 2 and 4 in Ref. 2 and those in Ref. 6. In those examples, the $\det[\hat{X}(t)]$ becomes zero if the trajectory is sufficiently long, in which case a conjugate point does exist, and the trajectory is nonoptimal. In the example treated here, the singularity of the matrix $X(t)$ in step 6 of the procedure does not result in the existence of a conjugate point.

Conclusions

A recent procedure for applying second-order necessary conditions and sufficient conditions is streamlined, described in detail, and illustrated by application to a published trajectory that satisfies only first-order necessary conditions for an optimal solution. The advantage of using this procedure over earlier methods is the ease with which the second-order conditions can be applied. In the multiple-terminal constraint problem analyzed here, in contrast to earlier examples, the first step of the two-step procedure indicates a possible conjugate point, but the second step of the procedure determines that no conjugate point exists.

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Queries

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