## 6.2 Legendre-Clebsch Necessary Condition

We have made use of the Legendre-Clebsch necessary condition in Step 4 of our Cookbook (in chapter 4). We now formally state and prove this condition through use of the Minimum Principle.

If $H$ is second-order differentiable in $\boldsymbol{u}$, the optimal control at $t$ is interior (not on the control boundary), and $\boldsymbol{u}(t)$ is "infinitesimally close" to $\boldsymbol{u}^*(t)$, i.e., a weak variation, then:

$$H^*_{\boldsymbol{u}}(t) = 0,\ H^*_{\boldsymbol{uu}}(t) \geq 0 \tag{6.49}$$

which means the matrix $H_{\boldsymbol{uu}}$ must be positive semi-definite.

The proof follows from the Minimum Principle:

$$H^*[t, \boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \boldsymbol{\lambda}(t)] \leq H[t, \boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\lambda}(t)] \tag{6.50}$$

Using a Taylor series to expand about $\boldsymbol{u}^*$ to second-order in $\boldsymbol{u}$ we obtain:

$$H^* \leq H^* + H^*_{\boldsymbol{u}}(\boldsymbol{u} - \boldsymbol{u}^*) + \frac{1}{2}(\boldsymbol{u} - \boldsymbol{u}^*)^T H^*_{\boldsymbol{uu}}(\boldsymbol{u} - \boldsymbol{u}^*) + \mathcal{O}(\Delta \boldsymbol{u}^3) \tag{6.51}$$

but $H^*_{\boldsymbol{u}} = 0$, from the Euler-Lagrange theorem since we assume the optimal control is not on the bound. Rearranging Eq. (6.51) and ignoring $\mathcal{O}(\Delta \boldsymbol{u}^3)$ we deduce using Eq.(6.50)

$$H^*_{\boldsymbol{uu}} \geq 0. \tag{6.52}$$

which is the Legendre-Clebsch necessary condition.

## 6.3 Notes on Necessary and Sufficient Conditions

**Example 6.3** Necessary and sufficient condition problem.

Consider Fig. 6.3. Suppose we have a set $A$ with the property:

$$\{(x, y) \in A\}. \tag{6.53}$$

A sufficient condition guarantees that $(x, y)$ is in the set $A$ but is usually too restrictive to uniquely define all $(x, y)$ in set $A$. A necessary condition applies to every element in the set $A$ but is usually too loose to define all $(x, y)$ in $A$ uniquely.

Now let us consider particular statements that pertain to the set $A$ in Fig. 6.3. For example:

$$x = \frac{1}{2}a,\ y = \frac{1}{2}b \tag{6.54}$$

Equations (6.54) provide a sufficient condition since they guarantee that $(x, y) \in A$. On the other hand:

$$0 \leq x \leq a \tag{6.55}$$

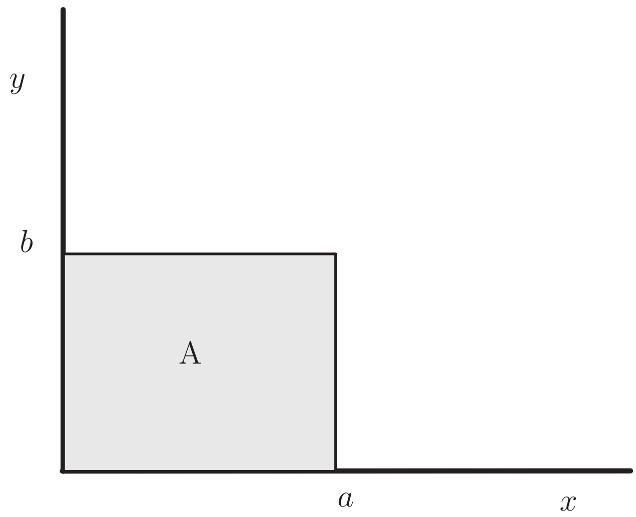


Figure 6.3: $(x, y) \in A$.

is a necessary but not sufficient condition that $(x, y) \in A$. Similarly

$$0 \leq y \leq b \tag{6.56}$$

is a necessary condition but it is not sufficient. Finally, we see that

$$0 \leq x \leq a,\ 0 \leq y \leq b \tag{6.57}$$

is a necessary and sufficient condition.

We note that the combination of two necessary conditions [Eqs. (6.55) and (6.56)] give a necessary and sufficient condition [Eq. (6.57)]. We also note that the combination of a necessary condition [say Eq. (6.55)] and a sufficient condition [say Eq. (6.54)] might not even come close to providing a necessary and sufficient condition.

Following Hull [2003], we can generalize the concepts of necessary conditions and sufficient conditions. Let $A$ and $B$ be mathematical statements and let the symbol "$\Rightarrow$" mean "implies" and the symbol "$\Leftarrow$" mean "is implied by". We can then write

$A \Rightarrow B$ means $A$ is sufficient for $B$

$A \Leftarrow B$ means that $A$ is necessary for $B$

$A \Leftrightarrow B$ means that $A$ is necessary and sufficient for $B$

For example

rain $\Rightarrow$ clouds

clouds $\Leftarrow$ rain

So, if it is raining then this statement is sufficient to know that it is cloudy. We can also say that a cloudy day is necessary for rain, but not sufficient.